Dynamic Feedback Linearization of Nonlinear Control Systems on Homogenous Time Scale

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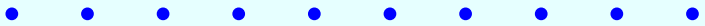
Time scale is a model of time

Definition

A **time scale** $\mathbb{T}$ is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers.

$\mathbb{T} = \mathbb{R}$ continuous time

$\mathbb{T} = \mathbb{Z}$ discrete time



$\mathbb{T} = \tau\mathbb{Z} := \{\tau k \mid k \in \mathbb{Z}\}, \ \tau > 0$ discrete time

$$\mathbb{T} = \mathbb{P}_{a,b} := \bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+a]$$

$$\mathbb{T} = \mathbb{H} := \left\{ 0, \sum_{k=1}^{n} \frac{1}{k} \mid n \in \mathbb{N} \right\}$$

- The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf \{\tau \in \mathbb{T} \mid \tau > t\}.$$

- The backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \inf\{\tau \in \mathbb{T} \mid \tau > t\}.$$

- The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

A time scale $\mathbb{T}$ is called homogeneous if $\mu \equiv \mathrm{const}.$

Definition

Delta derivative of $f(t) : \mathbb{T} \to \mathbb{R}$, denoted by $f^{\Delta}(t)$, can be defined as the extension of standard time-derivative in the continuous-time case.

time scale	$f^{\Delta}(t)$	delta derivative
$\mathbb{T} = \mathbb{R}$	$\frac{\mathrm{d}f(t)}{\mathrm{d}t}$	time derivative
$\mathbb{T} = \tau\mathbb{Z},\ \tau > 0$	$\dfrac{f(t+\tau) - f(t)}{\tau}$	difference operator

Consider a multi-input nonlinear dynamical system, defined on homogeneous time scale $\mathbb{T}$ and described by the state equations

$$x^{\Delta} = f(x, u), \tag{1}$$

where

- $x : \mathbb{T} \to \mathbb{X} \subset \mathbb{R}^n$ is an n-dimensional state vector;
- $u : \mathbb{T} \to \mathbb{U} \subset \mathbb{R}^m$ is an m-dimensional input vector;
- $f : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$ is assumed to be real analytic function.

- $\mathcal{K}$ is the field of meromorphic functions in a finite number of the independent system variables from the set

$$\mathcal{C} = \left\{ x_1, \dots, x_n;\ u_1^{\langle k \rangle}, \dots, u_m^{\langle k \rangle}, k \geq 0 \right\}.$$

- The pair $(\mathcal{K}, \sigma_f)$ is a σ_f-differential field.

- $\mathcal{K}^*$ denotes the inversive closure of $\mathcal{K}$.

- Consider the infinite set of symbols $\mathrm{d}\mathcal{C}^* = \{\mathrm{d}\zeta_i, \zeta \in \mathcal{C}^*\}$ and define by $\mathcal{E} := \mathrm{span}_{\mathcal{K}^*} \mathrm{d}\mathcal{C}^*$ the vector space spanned over the field $\mathcal{K}^*$ with

$$\mathcal{C}^* = \begin{cases} \mathcal{C}, & \text{if } \mu = 0, \\ \mathcal{C} \cup \left\{ z^{\langle -\ell \rangle} \mid \ell \geq 1 \right\}, & \text{if } \mu \neq 0. \end{cases}$$

- Any element of $\mathcal{E}$ is called differential one-form.

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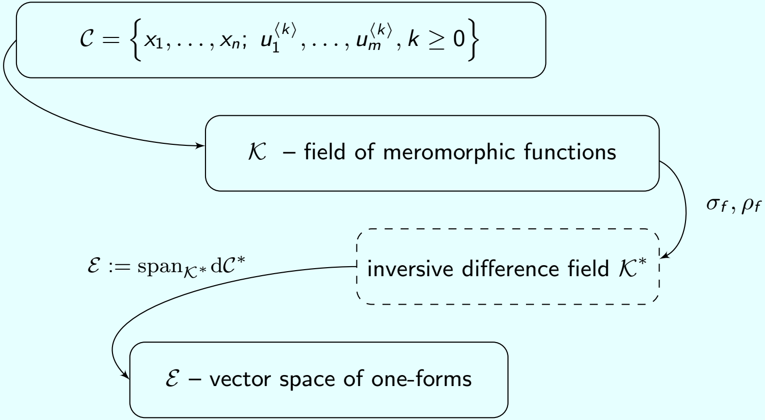

$\mathcal{C} = \left\{x_1, \ldots, x_n;\ u_1^{\langle k \rangle}, \ldots, u_m^{\langle k \rangle}, k \geq 0\right\}$
$\mathcal{K}$ – field of meromorphic functions
σ_f, ρ_f
inversive difference field $\mathcal{K}^*$
$\mathcal{E} := \mathrm{span}_{\mathcal{K}^*} \mathrm{d}\mathcal{C}^*$
$\mathcal{E}$ – vector space of one-forms

A left polynomial can be uniquely written in the form $\pi(z) = \sum_{\ell=0}^{k} \pi_\ell z^\ell$, $\pi_\ell \in \mathcal{K}^*$.

Definition

The skew polynomial ring, induced by σ_f-differential overfield $\mathcal{K}^*$, is the non-commutative ring $\mathcal{K}^*[z;\sigma_f,\Delta_f]$ of left polynomials in z with usual addition and multiplication satisfying, for any $\zeta \in \mathcal{K}^* \subset \mathcal{K}^*[z;\sigma_f,\Delta_f]$, the commutation rule

$$z\zeta := \zeta^{\sigma_f} z + \zeta^{\Delta_f}.$$

Let $\mathcal{K}^*[z;\sigma_f,\Delta_f]^{q\times q}$ denote the set of $q \times q$ polynomial matrices with entries in $\mathcal{K}^*[z;\sigma_f,\Delta_f]$.

Definition

A matrix $U(z) \in \mathcal{K}^*[z;\sigma_f,\Delta_f]^{q\times q}$ is called unimodular if there exists an inverse matrix $U^{-1}(z) \in \mathcal{K}^*[z;\sigma_f,\Delta_f]^{q\times q}$.

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A sequence of subspaces $\mathcal{H}_0 \supset \cdots \supset \mathcal{H}_{k^*} \supset \mathcal{H}_{k^*+1} = \mathcal{H}_{k^*+2} = \cdots =: \mathcal{H}_\infty$ of $\mathcal{E}$ is defined by

$$\mathcal{H}_0 := \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}x, \mathrm{d}u\},$$
$$\mathcal{H}_k := \left\{\omega \in \mathcal{H}_{k-1} \mid \omega^{\Delta_f} \in \mathcal{H}_{k-1}\right\}, \quad k \geq 1.$$

The sequence plays a key role in the analysis of various structural properties of nonlinear systems, including accessibility and feedback linearization.

Consider system (1) and suppose that the output function $y = h(x)$, $y \in \mathbb{Y} \subset \mathbb{R}^m$ is given. Define a chain of subspaces $\mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n$ of $\mathcal{E}$ as

$$\mathcal{E}_k = \operatorname{span}_{\mathcal{K}^*}\left\{\mathrm{d}x, \mathrm{d}y, \mathrm{d}y^{\langle 1\rangle}, \ldots, \mathrm{d}y^{\langle k\rangle}\right\}$$

and associated list of dimensions $p_k := \dim_{\mathcal{K}^*} \mathcal{E}_k$.

- For $k = 0, \ldots, n$, $\varsigma_k := p_k - p_{k-1}$ is the number of zeros at infinity of order less than or equal to k, with the convention $p_{-1} := n$.
- The rank p^* of the system is the total number of zeros at infinity, i.e., $p^* = \varsigma_n = p_n - p_{n-1}$.
- System (1) is said to be invertible if $p^* = m$.

Remark

The structure at infinity can be expressed in different manners. For instance, the list $\{n'_1, \ldots, n'_{p^*}\}$ of the orders of the zeros at infinity is the list of integers k such that $\varsigma_k - \varsigma_{k-1} \neq 0$, each one repeated $\varsigma_k - \varsigma_{k-1}$ times.

- **Static state feedback linearization**
- Dynamic state feedback linearization

Definition

The Brunovsky (controller) canonical form of a system (1), defined on time scale, is introduced as

$$
\begin{array}{ccc}
\xi_1^\Delta = \xi_2 & \cdots & \xi_{r_{m-1}+1}^\Delta = \xi_{r_{m-1}+2} \\
\xi_2^\Delta = \xi_3 & \cdots & \xi_{r_{m-1}+2}^\Delta = \xi_{r_{m-1}+3} \\
\vdots & & \vdots \\
\xi_{r_1-1}^\Delta = \xi_{r_1} & \cdots & \xi_{r_m-1}^\Delta = \xi_{r_m} \\
\xi_{r_1}^\Delta = v_1 & \cdots & \xi_{r_m}^\Delta = v_m
\end{array}
$$

with $r_1 + \cdots + r_m = n$ and $r_m \leq \cdots \leq r_2 \leq r_1$.

Note that $v : \mathbb{T} \to \mathbb{V} \subset \mathbb{R}^m$ is a vector of new inputs.

Theorem

Suppose $\mathcal{H}_\infty = \{0\}$. Then, there exists a list of integers $r_1, \ldots, r_m$ and m one-forms $\omega_1, \ldots, \omega_m \in \mathcal{H}_1$ whose relative degrees are, respectively, $r_1, \ldots, r_m$ such that

- $\mathrm{span}_{\mathcal{K}^*}\left\{\omega_i^{\Delta_f^j},\ i = 1, \ldots, m, j = 0, \ldots, r_j - 1\right\} = \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}x\} = \mathcal{H}_1$;
- $\mathrm{span}_{\mathcal{K}^*}\left\{\omega_i^{\Delta_f^j},\ i = 1, \ldots, m, j = 0, \ldots, r_j\right\} = \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}x, \mathrm{d}u\} = \mathcal{H}_0$;
- *the one-forms* $\left\{\omega_i^{\Delta_f^j},\ i = 1, \ldots, m, j \geq 0\right\}$ *are linearly independent; in particular*
  $$\sum_{i=1}^{m} r_i = n.$$

Theorem

System (1) is linearizable by regular[a] static state feedback $u = \psi(x, v)$ iff $\mathcal{H}_\infty = \{0\}$ and $\mathcal{H}_k$, for $k = 1, \ldots, k^*$, are integrable.

[a] A compensator is called regular, if it is invertible, i.e., $\mathrm{rank}_{\mathcal{K}^*} \dfrac{\partial \psi}{\partial v} = m$.

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Suppose $\mathcal{H}_\infty = \{0\}$. *Then, there exists a list of integers* $r_1, \ldots, r_m$ *and* m *one-forms* $\omega_1, \ldots, \omega_m \in \mathcal{H}_1$ *whose relative degrees are, respectively,* $r_1, \ldots, r_m$ *such that*

- $\mathrm{span}_{\mathcal{K}^*}\left\{\omega_i^{\Delta_f^j},\ i = 1, \ldots, m, j = 0, \ldots, r_j - 1\right\} = \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}x\} = \mathcal{H}_1$;
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- Static state feedback linearization
- **Dynamic state feedback linearization**

System (1) is said to be linearizable by dynamic state feedback if there exist a regular dynamic compensator of the form

$$\begin{aligned} \eta^{\Delta} &= \zeta(x, \eta, v), \\ u &= \psi(x, \eta, v) \end{aligned} \tag{2}$$

with $\eta \in \mathbb{R}^s$, and an extended coordinate transformation $\xi = \phi(x, \eta)$ such that, in the new coordinates, the compensated system (1) reads as

$$\xi^{\Delta} = A\xi + Bv,$$

where $\xi \in \mathbb{R}^{n+s}$ and the pair (A, B) is in Brunovsky canonical form.

Define the subspaces of $\mathcal{E}$ as $\mathcal{X} := \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}x\}, \mathcal{Y} := \mathrm{span}_{\mathcal{K}^*}\left\{\mathrm{d}y^{\langle k \rangle}, k \geq 0\right\}, \mathcal{X}_\nu := \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}x, \mathrm{d}u, \mathrm{d}u^{\langle 1 \rangle}, \dots, \mathrm{d}u^{\langle \nu-1 \rangle}\}$.

Definition

A linearizing output is an output function $y = h\left(x, u, u^{\langle 1 \rangle}, \dots, u^{\langle \nu-1 \rangle}\right)$ that satisfies the following properties:

- $y = h\left(x, u, u^{\langle 1 \rangle}, \dots, u^{\langle \nu-1 \rangle}\right)$ defines an invertible system;
- $\sum_i n_i' = \dim_{\mathcal{K}^*}(\mathcal{X} \cap \mathcal{Y}) = n$.

Theorem

Suppose $\mathcal{H}_\infty = \{0\}$, *and let* $\Omega := \begin{bmatrix}\omega_1 & \ldots & \omega_m\end{bmatrix}^{\mathrm{T}} \in \mathcal{E}^m$ *be a system of linearizing one-forms for system (1). Then, there exists a system of linearizing outputs iff there exists a unimodular polynomial matrix* $U(z) \in \mathcal{K}^*[z; \sigma_f, \Delta_f]^{m\times m}$ *such that*

$$\mathrm{d}(U(z)\Omega) = 0.$$

Corollary

Let (1) be a single-input system and suppose $\mathcal{H}_\infty = \{0\}$. *Then, the following statements are equivalent:*

- *(1) is linearizable by static state feedback;*
- *(1) is linearizable by dynamic state feedback;*
- $\mathrm{d}\omega_1 \wedge \omega_1 = 0$, *where* ω_1 *is such that* $\mathcal{H}_n = \mathrm{span}_{\mathcal{K}^*}\{\omega_1\}$.

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Consider the system

$$\begin{aligned} x_1^{\Delta} &= x_2 - u_1 \\ x_2^{\Delta} &= x_4 u_1 \\ x_3^{\Delta} &= u_1 \\ x_4^{\Delta} &= u_2. \end{aligned} \tag{3}$$

The sequence of subspaces $\mathcal{H}_k$, $k \geq 0$ can be calculated as

$$\begin{aligned} \mathcal{H}_1 &= \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3, \mathrm{d}x_4\}, \\ \mathcal{H}_2 &= \mathrm{span}_{\mathcal{K}^*}\{x_4^{\rho_f}\mathrm{d}x_1 + \mathrm{d}x_2, \mathrm{d}x_1 + \mathrm{d}x_3\}, \\ \mathcal{H}_3 &= \cdots = \mathcal{H}_\infty = \{0\}. \end{aligned}$$

For this example both linearizing one-forms can be chosen from $\mathcal{H}_2$, i.e., $\Omega := \begin{bmatrix}\omega_1 & \omega_2\end{bmatrix}^{\mathrm{T}}$, where $\omega_1 = x_4^{\rho_f}\mathrm{d}x_1 + \mathrm{d}x_2$ and $\omega_2 = \mathrm{d}x_1 + \mathrm{d}x_3$. Though $\mathcal{H}_\infty = \{0\}$, the system is not linearizable by static state feedback, since $\mathrm{d}\omega_1 \wedge \omega_1 \wedge \omega_2 = -\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4^{\rho_f} \neq 0$.

However, the system is linearizable by dynamic state feedback. Indeed, take

$$U(z) = \begin{bmatrix} \frac{1}{x_4^{\rho_f}} & -\frac{1}{x_4^{\rho_f}} z \\ 0 & 1 \end{bmatrix}$$

for which the inverse matrix can be found as

$$U^{-1}(z) = \begin{bmatrix} x_4^{\rho_f} & z \\ 0 & 1 \end{bmatrix}$$

Next, verify that

$$U(z)\Omega = \begin{bmatrix} \mathrm{d}x_1 \\ \mathrm{d}(x_1 + x_3) \end{bmatrix}.$$

Hence, the linearizing outputs are $y_1 = x_1$ and $y_2 = x_1 + x_3$. Next, compute the sequence of subspaces $\mathcal{E}_k$ for $k = 0, \dots, 4$ as

$$\begin{aligned}
\mathcal{E}_0 &= \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}x\}, \\
\mathcal{E}_1 &= \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}x, -\mathrm{d}u_1\}, \\
\mathcal{E}_2 &= \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}x, -\mathrm{d}u_1, -\mathrm{d}u_1^{\Delta}\}, \\
\mathcal{E}_3 &= \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}x, -\mathrm{d}u_1, -\mathrm{d}u_1^{\Delta}, -\mathrm{d}u_1^{\langle 2\rangle}, \lambda_1 \mathrm{d}u_2\}, \\
\mathcal{E}_4 &= \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}x, -\mathrm{d}u_1, -\mathrm{d}u_1^{\Delta}, -\mathrm{d}u_1^{\langle 2\rangle}, -\mathrm{d}u_1^{\langle 3\rangle}, \lambda_1 \mathrm{d}u_2, \lambda_2 \mathrm{d}u_2^{\Delta}\},
\end{aligned}$$

where $\lambda_1, \lambda_2 \in \mathcal{K}^*$. Hence, it follows that $p = \{4, 5, 6, 8, 10\}$, and therefore, $\varsigma = \{0, 1, 1, 2, 2\}$. Thus, we may conclude that the system is invertible, since $p^* = \varsigma_4 = 2$. From computations of the subspaces $\mathcal{E}_k$, we know that

$$\begin{aligned}
y_1^{\Delta} &= x_2 - u_1 \\
y_2^{\langle 3\rangle} &= (x_4 + \mu u_2)\left(x_4 u_1 - y_1^{\langle 2\rangle}\right) + u_2(x_2 - y_1^{\Delta}).
\end{aligned}$$

Take $\eta = y_1^{\Delta}$, $\eta^{\Delta} = v_1$, and $y_2^{\langle 3 \rangle} = v_2$ then the dynamic feedback compensator has the form

$$\begin{aligned} \eta^{\Delta} &= v_1 \\ u_1 &= x_2 - \eta \\ u_2 &= \frac{v_2 - x_4(x_4(x_2 - \eta) - v_1)}{\mu(x_4(x_2 - \eta) - v_1) + x_2 - \eta}. \end{aligned} \tag{4}$$

Now, relying on the inversion algorithm we can calculate dimension of the extended state equations according to the formula $s = \sum_{i=1}^{m} (\epsilon_i - \gamma_i)$ as $s = (2-1) + (3-3) = 1$. The application of (4) to system (3) yields the extended state equations

$$\begin{aligned} x_1^{\Delta} &= \eta \\ x_2^{\Delta} &= x_4(x_2 - \eta) \\ x_3^{\Delta} &= x_2 - \eta \\ x_4^{\Delta} &= \frac{v_2 - x_4(x_4(x_2 - \eta) - v_1)}{\mu(x_4(x_2 - \eta) - v_1) + x_2 - \eta} \\ \zeta^{\Delta} &= v_1 \end{aligned} \tag{5}$$

Then we define the coordinate transformation as

$$\begin{aligned}
\xi_1 &:= y_1 = x_1 \\
\xi_2 &:= y_1^{\Delta} = x_1^{\Delta} = x_2 - u_1 \\
\xi_3 &:= y_2 = x_1 + x_3 \\
\xi_4 &:= y_2^{\Delta} = x_2 - u_1 + u_1 = x_2 \\
\xi_5 &:= y_2^{\langle 2 \rangle} = x_4 u_1.
\end{aligned}$$

In the new coordinates the extended system has the linear form

$$\xi_1^{\Delta} = \xi_2 \qquad \xi_2^{\Delta} = v_1 \qquad \xi_3^{\Delta} = \xi_4$$
$$\xi_4^{\Delta} = \xi_5 \qquad \xi_5^{\Delta} = v_2.$$

Thank you very much for your attention!

Any questions?