Optimal Control of Lossy Energy Storage Systems with Nonlinear Efficiency Based on Dynamic Programming and Pontryagin’s Minimum Principle

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Abstract—We consider energy storage systems having nonlinear efficiency functions, which are becoming increasingly important as shown in several recent works, and propose an optimal solution based on Pontryagin’s minimum principle. A central challenge in such problems is the hard limits on the state variable, which restrict the use of the minimum principle. To address this challenge, we propose to include the capacity constraints in the objective function with a proper weighting constant. We show that this approach allows formulation of the problem based on the classical minimum principle, and eventually leads to an efficient optimal control strategy. The proposed solution is compared to a dynamic programming algorithm. Numeric experiments reveal that for lossless storage devices dynamic programming is beneficial, since it enables fast and accurate solutions when a low number of samples is used. However, for lossy storage devices the situation is the opposite, and the minimum principle provides faster and more accurate solutions, since its computational complexity is almost unaffected by changes in the system parameters.

I. INTRODUCTION

ENERGY storage optimal control problems are typically nonconvex, and therefore hard to solve due to their high numeric complexity [1]. Several solutions to this problem have been suggested in the recent literature, among them Pontryagin’s minimum principle, which is gaining attention due to its analytic nature. The minimum principle is an established approach in optimal control theory, where first-order necessary conditions for optimality are evaluated for constrained dynamical systems [2, Chapter 6]. Although the minimum principle is not straightforward to use, its main advantage is that it may provide simple solutions with low complexity, which can be implemented in real time [3]. In the context of storage devices, results based on the minimum principle can be categorized into two major sub-classes: contributions that focus on power grids are highlighted in [4]–[10], whereas studies centered around hybrid electrical vehicles are witnessed in [3], [11]–[22]. The former mostly consider electrical storage devices, while works in [4], [5] solely focus on thermal storage devices. Typical objectives of control policy optimization are to minimize the fuel consumption [10], [13], [22] or the power consumption [6], [9], while several other optimal controllers are proposed to maximize the revenue from energy trading [4], [5]. The last work introduces a bang-off-bang optimal policy for a capacitor type storage device. It considers natural constraints on voltage and current, and describes a usage-dependent aging rate of the storage device. More recently paper [10] revisits the “shortest-path” method presented in [23], [24], and shows that this path can be derived explicitly by means of the minimum principle. The solution in [10] is analytic in nature, and is used to design a low-complexity algorithm for calculating the shortest path. Most of the recent studies, in particular [3], [6], [9], [10], [14], [22], consider batteries due to their falling costs, and define their charge as the state variable, which is often bounded within a range. Other constraints may include limits on the power losses, and in vehicles, the torque and the angular velocity as highlighted in [3], [11], [20]. In contrast to problems with single storage devices, the combination of batteries and super-capacitors is explored in [9], [11], [13], [22] to enhance the overall efficiency and to delay battery degradation. Moreover, paper [9] demonstrates that this combination is useful for multiple grid services: the batteries provide peak shaving, while the capacitor compensates the short-term forecasting errors, and facilitates high-frequency power fluctuations. Likewise, paper [6] illustrates that the combination of different storage devices typically reduces the fuel consumption compared to single storage systems.

Another issue which might complicate the optimal solution is losses. Several works such as [7], [10] assume that the system is ideal and lossless, while other works either consider storage devices as having constant efficiency [4]–[6], [9], or else they model losses as resistance [20] or use look-up tables [13], [22]. Optimal control policies for thermal storage devices with losses are presented in [4], [5]. The objective in these works is maximal revenue, considering bounds on the power flow and the device capacity. In [4] it is assumed that the stored energy is unconstrained, and this assumption is relaxed in [5] by considering bounds on the stored energy. In addition, continuous-time optimal control methods are presented in [6], [9] to solve a power management problem in microgrids having centralized [9] and distributed storage devices [6] with constant losses. The objectives of the controllers are to minimize the power exchange between the main grid (e.g.,
using dynamic programming, and then presents a comparative
Section IV first resolves the problem addressed in Section III
perform a series of numerical simulations on electric vehicles
whereas for lossy storage devices the minimum principle
ming approach yields fast and relatively accurate solutions,
programming strategies [27, Chapter
This is done by comparing the minimum principle to dynamic
control algorithm for both the "lossless" and the "lossy" cases.
We further provide guidelines for choosing the best optimal
control policy that maximizes the achievable regeneration efficiency.
The results above solely target devices having constant
losses, and an extension of the minimum principle to address
time-varying efficiency functions is currently unavailable. In
light of this gap, in this article we consider storage devices
having nonlinear efficiency functions, which are becoming
increasingly important as shown in several recent works (e.g.,
[25] and [26]), and propose an optimal solution based on
Pontryagin’s minimum principle. We show that this general-
ized nonlinear efficiency function and the finite capacity of
the storage device result in a challenging control problem.
The first complication originates from the nonlinear efficiency
in the device dynamics: since the resulting Hamiltonian is
difficult to differentiate, an explicit control relation cannot
be directly established. In addition, the finite capacity of the
storage device typically introduces hard limits on the state
variable, which also restricts use of the minimum principle. To
address these challenges, we propose a modified value function
which incorporates the constraints on the state variable. This
modification is mainly done by adding an appropriate dead-
zone function to the objective. Subject to these constraints and
considerations we provide an explicit solution, and calculate
the optimal control policy.
We further provide guidelines for choosing the best optimal
control algorithm for both the “lossless” and the “lossy” cases.
This is done by comparing the minimum principle to dynamic
programming strategies [27, Chapter 2]. Both solutions are
tested based on the Ashalim molten salt storage device, which
was recently installed as part of the Israeli power network. We
show that for lossless storage devices the dynamic program-
ning approach yields fast and relatively accurate solutions,
whereas for lossy storage devices the minimum principle
approach provides better results (see Section IV-B). Finally, we
perform a series of numerical simulations on electric vehicles
to verify the applicability of the proposed control methods.

The rest of the paper is organized as follows. Section II
formulates the problem and provides the background for the
developed results. Section III proposes an optimal control pol-
icy for lossy storage devices based on the minimum principle.
Section IV first resolves the problem addressed in Section III
using dynamic programming, and then presents a comparative
study among the control policies developed in Sections III
and IV. Concluding remarks are given in Section V.

Notations: The notations employed here are standard: \( \mathbb{R}_{\geq 0} \)
(resp. \( \mathbb{R}_{> 0} \)) represents the set of non-negative (resp. positive)
real numbers. For a function \( f : \mathcal{X} \rightarrow \mathcal{Y} \), \( f'() \) and \( f''() \)
denote its first and second derivative, while \( f^{-1} \) denotes its inverse. In addition, the ceiling function is denoted as \( \lceil \cdot \rceil \).

II. TECHNICAL PRELIMINARIES

We consider a grid-connected storage device with losses. The
device is charged from the grid and feeds an aggregated
load, which is characterized by its active power consumption.
In contrast to the stochastic loads shown in [28], [29], we
consider here the load power \( P_L : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R} \), which is a
(positive) continuous function, and its consumption over the
time interval \([0,T]\) for some arbitrary \( T > 0 \) is known. The storage device is charged or discharged by a net generation
process associated with the rate \( u(t) = P_g(t) - P_L(t) \), where
\( P_g : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R} \) and \( u : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R} \) denote the grid’s power
and the storage device power, respectively. In addition, \( E_0(t) \),
\( E_L(t) \), and \( x(t) \) denote the generated energy, load energy,
and the stored energy, respectively. Based on these terminologies,
the model is
\[
\frac{d}{dt} x(t) = B u(t),
\]
where \( x(t) = [E_g(t) \ E_L(t) \ x(t)]^T \) and \( u(t) = [P_g(t) \ P_L(t) \ u(t)]^T \). The matrix \( B \) is calculated as
\[
B = \begin{bmatrix}
0 & 1 & 1 \\
0 & 1 & 0 \\
0 & 0 & \eta(u(t))
\end{bmatrix},
\]
where \( \eta : \mathbb{R} \rightarrow (0,1] \) is a continuous function, which
characterizes the efficiency of the storage device.

The power \( P_g(t) \) is controllable, and is characterized by a
fuel consumption function \( F(P_g(t)) \). The function
\( F(P_g(t)) \in \mathbb{R}_{\geq 0} \) represents the fuel consumption at time
\( t \geq 0 \), and in a broader sense, refers to a cost function
associated with power generation. This function is assumed to
be twice differentiable and \( F''(P_g(t)) > 0 \) [30]. Following
these definitions, the total cost is
\[
F_{\text{total}} := \int_0^T F(P_g(t)) dt.
\]

In general, dynamic models of storage devices are based
either on the device physics or on energy flow models as
described in [31]. In light of (1), we consider a generalized
non-affine dynamic model of the storage device, given as
\[
\frac{d}{dt} x(t) = \eta(u(t)) u(t).
\]
Here \( \eta(u(t)) \) typically accounts for the device losses and can
be defined as
\[
\eta(u(t)) = \begin{cases} 
\eta_c(u(t)), & \text{if } u(t) \geq 0, \\
\eta_d^{-1}(u(t)), & \text{otherwise},
\end{cases}
\]
where \( \eta_c(u(t)) \) and \( \eta_d(u(t)) \) denote the charging and the dis-
charging efficiencies, respectively. Both \( \eta_c(u(t)) \) and \( \eta_d(u(t)) \)
are positive functions, and take values in the range \((0,1]\). It
is further assumed that the device capacity is \( E_{\text{max}} > 0 \).
Therefore, the device instantaneous energy \( x(t) \) is bounded by
\( x(t) \in [0, E_{\text{max}}] \).

Since \( F''(P_g(t)) > 0 \), it implies \( F(P_g(t)) \) is a strictly convex function.
A. Problem Formulation

Since \( P_g(t) \) can be controlled, the key challenge is to determine the optimal generated power \( P_g(t) \) at every point in time, such that the total cost \( F_{\text{total}}(t) \) in (2) is minimized. This leads to the following optimal control problem:

\[
\begin{align*}
\text{minimize} & \quad F_{\text{total}} = \int_0^T F(P_g(t)) \, dt, \\
\text{subject to} & \quad \frac{dx(t)}{dt} = \eta(u(t))u(t), \quad x(T) = 0, \\
& \quad 0 \leq x(t) \leq E_{\text{max}}, \quad u(t) = P_g(t) - P_L(t),
\end{align*}
\]

which can be further simplified by replacing \( P_g(t) = u(t) + P_L(t) \), as

\[
\begin{align*}
\text{minimize} & \quad F_{\text{total}} = \int_0^T F(u(t) + P_L(t)) \, dt, \\
\text{subject to} & \quad \frac{dx(t)}{dt} = \eta(u(t))u(t), \quad x(0) = 0, \\
& \quad 0 \leq x(t) \leq E_{\text{max}}, \quad x(T) = 0.
\end{align*}
\]

In the subsequent sections we solve (4) and evaluate an optimal \( u(t) \); thus, the optimal \( P_g(t) \) can be obtained accordingly.

B. Optimal Control Policy for Lossless Storage

In this section we give a succinct illustration of the optimal control policy for lossless storage devices. First, we reformulate the problem (4) employing \( \eta(u(t)) = 1 \). Since Pontryagin’s minimum principle does not include hard limits on the state variable, we define a modified running cost function which incorporates the constraint on \( x(t) \) as

\[
J = \int_0^T \left[ F(u(t) + P_L(t)) + c(x(t)) \right] \, dt,
\]

where the function \( c : \mathbb{R} \to [0, \infty) \) is defined as

\[
c(x(t)) := \begin{cases} 
\frac{\rho}{2E_{\text{max}}} (x - E_{\text{max}})^2, & \text{for } x(t) > E_{\text{max}}, \\
0, & \text{for } 0 \leq x(t) \leq E_{\text{max}}, \\
\frac{\rho}{2E_{\text{max}}} \lambda^2, & \text{for } x(t) < 0,
\end{cases}
\]

with \( \rho > 0 \). If the constant \( \rho \) is very large, then \( x(t) \) is practically limited to the range \( 0 \leq x(t) \leq E_{\text{max}} \), since otherwise the resulting cost is high. Using the modified running cost in (5), problem (4) can be presented as

\[
\begin{align*}
\text{minimize} & \quad J = \int_0^T \left[ F(u(t) + P_L(t)) + c(x(t)) \right] \, dt, \\
\text{subject to} & \quad \frac{dx(t)}{dt} = u(t), \quad x(0) = 0, \quad x(T) = 0.
\end{align*}
\]

In the sequel, we intend to solve (7) employing the minimum principle. To this end, first we define the Hamiltonian \( \mathcal{H}(x(t), \lambda(t), u(t), t) \) of (7) as

\[
\mathcal{H}(x(t), \lambda(t), u(t), t) = F(u(t) + P_L(t)) + c(x(t)) + \lambda(t)u(t).
\]

Let \( t \to (\hat{x}(t), \hat{\lambda}(t), \hat{u}(t)) \) be the optimal solution of (7). Then this function obeys the following conditions

\[
\begin{align*}
(a1) & \quad \frac{d}{dt} \hat{x}(t) = \hat{u}(t), \quad \hat{x}(0) = 0, \quad \hat{x}(T) = 0, \\
(b1) & \quad \frac{d}{dt} \hat{\lambda}(t) = -c(\hat{x}(t)), \\
(c1) & \quad \frac{d}{dt} \left\{ F(u(t) + P_L(t)) + \hat{\lambda}(t)u(t) \right\} \bigg|_{u(t) = \hat{u}(t)} = 0,
\end{align*}
\]

where

\[
c'(\hat{x}(t)) = \begin{cases} 
\frac{\rho}{E_{\text{max}}} (\hat{x}(t) - E_{\text{max}}), & \text{for } \hat{x}(t) > E_{\text{max}}, \\
0, & \text{for } 0 \leq \hat{x}(t) \leq E_{\text{max}}, \\
\frac{\rho}{E_{\text{max}}} \hat{x}(t), & \text{for } \hat{x}(t) < 0.
\end{cases}
\]

Now replacing \( \hat{\lambda}(t) \) with \( -\hat{\lambda}(t) \) in (b1), and simplifying condition (c1), yields

\[
\begin{align*}
(a2) & \quad \frac{d}{dt} \hat{x}(t) = \hat{u}(t), \quad \hat{x}(0) = 0, \quad \hat{x}(T) = 0, \\
(b2) & \quad \frac{d}{dt} \hat{\lambda}(t) = c'(\hat{x}(t)), \\
(c2) & \quad \hat{u}(t) = (F')^{-1}(\hat{\lambda}) - P_L(t).
\end{align*}
\]

Since \( F'(\cdot) > 0 \), the inverse function \( (F')^{-1}(\cdot) \) in (c2) is well-defined. Then replacing (c2) in (a2) evaluates the following necessary optimal solutions

\[
\begin{align*}
(a3) & \quad \frac{d}{dt} \hat{x}(t) = \hat{u}(t) = (F')^{-1}(\hat{\lambda}) - P_L(t), \quad \hat{x}(0) = 0, \quad \hat{x}(T) = 0, \\
(b3) & \quad \frac{d}{dt} \hat{\lambda}(t) = c'(\hat{x}(t)).
\end{align*}
\]

Let \( \hat{E}_g(t) \) be the optimal generated energy. Then based on (a3) and (b3), an equivalent solution is obtained by substituting \( \hat{x}(t) = \hat{E}_g(t) - E_L(t) \) as

\[
\begin{align*}
(a4) & \quad \frac{d}{dt} \hat{E}_g(t) = \hat{u}(t) = (F')^{-1}(\hat{\lambda}), \quad \hat{E}_g(0) = E_L(0) = 0, \\
& \quad \hat{E}_g(T) = E_L(T); \\
(b4) & \quad \frac{d}{dt} \hat{\lambda}(t) = c'(\hat{x}(t)).
\end{align*}
\]

These are necessary conditions for an optimal solution, for the lossless storage device.

Considering the “shortest path” results documented in [24], [32], the following theorem shows that for \( g \to \infty \) the optimal generated energy shown in (a4) also follows the shortest path set by the load profile and the device capacity.

**Theorem 1** ([10, Theorem 1]). Recall the function \( c(x(t)) \) as in (6). Then, for \( g \to \infty \), the optimal solution \( \hat{E}_g(t) \) converges to the shortest path between \( E_L(t) \) and \( E_L(t) + E_{\text{max}} \).

A formal proof of the above theorem is documented in [10], hence omitted.

III. OPTIMAL CONTROL POLICY FOR LOSSY STORAGE DEVICES

In this section we solve (4) and characterize an optimal control policy for storage devices having time-varying losses. The main challenge is that in light of the time-varying losses, condition (c2) cannot be derived from (c1) in a straightforward fashion, due to the presence of the non-affine efficiency function \( \eta(u(t)) \) (see (11)). To solve this problem, in the following analysis we assume that for every optimal costate variable \( \hat{\lambda}(t) \) and \( P_L(t) \), there always exists a unique solution to the derivative of the Hamiltonian function (11). In this
regard, reconsidering the modified cost function in (5), we pose the subsequent optimal control problem to

\[
\begin{align*}
\text{minimize} \quad & J = \int_0^T \left[ F(u(t) + P_L(t)) + c(x(t)) \right] dt, \\
\text{subject to} \quad & \frac{d}{dt} x(t) = \eta(u(t)) u(t), \\
& x(0) = 0, \quad x(T) = 0.
\end{align*}
\] (10)

The following analysis solves (10) by means of the minimum principle. First, we compute the Hamiltonian \( \mathcal{H}(x(t), \lambda(t), u(t), t) \) for a lossy storage device having conditions for optimality.

In contrast to the results shown in Section II-B, conditions (A3) yields\n
\[
\begin{align*}
A1 \quad & \frac{d}{dt} \tilde{x}(t) = \eta(\tilde{u}(t)) \tilde{u}(t), \quad \tilde{x}(0) = 0, \quad \tilde{x}(T) = 0, \\
A2 \quad & \frac{d}{dt} \tilde{\lambda}(t) = -c'(\tilde{x}(t)), \\
A3 \quad & \frac{d}{dt} \left[ F(u(t) + P_L(t)) + \tilde{\lambda}(t)\eta(u(t))u(t) \right] |_{u(t) = \tilde{u}(t)} = 0,
\end{align*}
\]

where the \( c'(\tilde{x}(t)) \) stated in (A2), is the same as in (9). Now replacing \( \tilde{\lambda}(t) \) with \( -\tilde{\lambda}(t) \) in (A2), and simplifying condition (A3) yields\n
\[
\begin{align*}
B1 \quad & \frac{d}{dt} \hat{x}(t) = \eta(\hat{u}(t)) \hat{u}(t), \quad \hat{x}(0) = 0, \quad \hat{x}(T) = 0, \\
B2 \quad & \frac{d}{dt} \hat{\lambda}(t) = c'(\hat{x}(t)), \\
B3 \quad & F'(\hat{u}(t) + P_L(t)) = \lambda \frac{d}{dt} (\eta(\hat{u}) \hat{u}).
\end{align*}
\]

It is assumed that for every \( \lambda \) and \( P_L(t) \), condition (B3) has a single solution with respect to \( \hat{u}(t) \), which can be defined as \( \hat{u}(t) := \hat{u}(P_L(t), \lambda) \). Then, replacing \( \hat{u}(t) \) in (B1) yields\n
\[
\begin{align*}
C1 \quad & \frac{d}{dt} \hat{x}(t) = \eta(\hat{u}(P_L(t), \lambda)) \hat{u}(P_L(t), \lambda), \quad \hat{x}(0) = 0, \quad \hat{x}(T) = 0, \\
C2 \quad & \frac{d}{dt} \hat{\lambda}(t) = c'(\hat{x}(t)).
\end{align*}
\]

In contrast to the results shown in Section II-B, conditions (C1) and (C2) stated above illustrate necessary conditions for optimality.

These conditions can be also used to solve the optimal peak shaving problem (see [32]) for a lossy storage device having constant efficiency functions. In particular, we seek to resolve the optimization problem (4) in which the cost function is defined as

\[
F(P_g(t)) = \frac{1}{k+1} P_g(t)^{1+k} \quad \text{for} \quad k = 1, 3, 5, \ldots. \quad (12)
\]

The efficiency function is given in (3), and is depicted in Fig. 1. The reason for selecting the cost function stated in (12) is that a high value of \( k \) penalizes \( P_g(t) \). Therefore, with \( k \to \infty \) this cost function minimizes the generated power peak, which results in optimal peak shaving. The integer \( k \) is selected as an odd number to simplify the numeric calculations.

To solve the above problem, first in Fig. 2(a) we compute the approximate derivative \( \frac{d}{du}(\eta(u)\hat{u}) \) associated with the efficiency function in Fig. 1.

Fig. 1. A simple efficiency function \( \eta(u) \). Here \( \eta_c, \eta_d \) are the charging and the discharging efficiencies, respectively. Both are constants in the range \((0, 1)\).

Now employing the condition (C2), an optimal \( \hat{u}(t) \) may be obtained by finding the intersection of two curves; see Fig. 2(b). Note that, for every \( P_L(t) \geq 0 \) and for every \( \lambda \) these graphs intersect at a single point, hence condition (C2) has a single solution with respect to \( \hat{u}(t) \), as required. According to Fig. 2(b) an optimal \( \hat{u}(t) \) can be computed as

\[
\hat{u}(t) = \begin{cases} 
\left( \eta_c \lambda \right)^{\frac{1}{k}} - P_L(t), \quad \text{for} \quad \hat{u}(t) > 0, \\
\left( \eta_d^{-1} \lambda \right)^{\frac{1}{k}} - P_L(t), \quad \text{for} \quad \hat{u}(t) < 0, \\
0, \quad \text{otherwise.}
\end{cases}
\]

Since \( k \) is a positive and odd integer, the root of \( (\eta_c \lambda)^{\frac{1}{k}} \) is well-defined. Hence, for \( P_L(t) \geq 0 \), the explicit solution yields

\[
\begin{align*}
D1 \quad & \frac{d}{dt} \hat{x}(t) = \eta(\hat{u}(t)), \quad \hat{x}(0) = 0, \quad \hat{x}(T) = 0, \\
D2 \quad & \frac{d}{dt} \hat{\lambda}(t) = c'(\hat{x}(t)),
\end{align*}
\]

where \( c'(\hat{x}(t)) \) is denoted in (9), and \( \hat{u}(t) \) is defined as

\[
\hat{u}(t) = \begin{cases} 
\left( \eta_c \lambda \right)^{\frac{1}{k}} - P_L, \quad \text{for} \quad P_L < \left( \eta_c \lambda \right)^{\frac{1}{k}}, \\
\left( \eta_d^{-1} \lambda \right)^{\frac{1}{k}} - P_L, \quad \text{for} \quad P_L > \left( \eta_d^{-1} \lambda \right)^{\frac{1}{k}}, \\
0, \quad \text{otherwise.}
\end{cases}
\]

IV. COMPARATIVE ANALYSIS

This section provides a comparative analysis between two complementary optimal control strategies: 1) the proposed solution, based on the minimum principle and 2) dynamic programming. Some works that illustrate general interconnections between these solutions include [33]–[35]. In addition, papers [36]–[38] demonstrate comparative studies of these methods in the context of hybrid electric vehicles. However, for a lossy storage device integrated into a general power grid
it is still unclear which algorithm should be used under which circumstances. To answer this question, first in Section IV-A we resolve the optimal control problem (7) and (10) employing a dynamic programming algorithm. Then, in Section IV-B we numerically compare the results obtained in Section IV-A with the optimal solutions presented in Section II-B and III. To conduct the numerical experiments first we consider the Ashalim storage system which is a part of the Israeli power grid, and is situated in the Negev Desert in Israel. Then Section IV-C shows additional results for electric vehicles.

A. Solution based on Dynamic Programming

This section briefly explains how a dynamic programming-based strategy may be utilized to resolve (7) and (10). Here, we only solve (10). The solution of (7) is trivial, and can be obtained considering \( \eta(u(t)) = 1 \) in the following algorithm.

Using dynamic programming, the primary optimization problem is divided into a series of simpler subproblems, and a global optimal solution is obtained by scanning all feasible solutions iteratively; see [27, Chapter 2] for the details. Dynamic programming algorithms often work in discrete-time. Hence, we first convert the continuous-time problem to its equivalent discrete-time counterpart. Considering this, the optimization problem (4) is formulated in discrete-time as

\[
\begin{align*}
\text{minimize} & \quad \Delta \sum_{i=1}^{N} F(u_i + P_{L,i}), \\
\text{subject to} & \quad x_k = \Delta \sum_{i=1}^{k} \eta(u_i) u_i, \quad x(0) = 0, \quad 0 \leq x_k \leq E_{\text{max}}, \quad x(T) = 0, \\
& \quad k = 1, \ldots, N,
\end{align*}
\]

where \( T > 0 \) defines the final time, \( N \) is the number of samples, and \( \Delta = \left[ \frac{T}{N} \right] \). To solve (14), the following cost function is considered

\[
V_k(x_k) := \Delta \min_{u_1, \ldots, u_k} \sum_{i=1}^{k} F(u_i + P_{L,i}),
\]

in which, using (14), \( V_1(x_1) \) is calculated as

\[
V_1(x_1) = \Delta F(u_1 + P_{L,1}) = \Delta F \left( \frac{x_1}{\eta(u_1)} \Delta + P_{L,1} \right),
\]

while for \( k = 2, \ldots, N, \)

\[
V_k(x_k) = \min_{0 \leq x_{k-1} \leq E_{\text{max}}} \left\{ V_{k-1}(x_{k-1}) + \Delta F \left( \frac{x_{k-1}}{\eta(u_{k-1})} \Delta + P_{L,k} \right) \right\},
\]

where \( x_k \in [0, E_{\text{max}}] \). Now, after computing the values of \( V_k(x_k) \) for every \( k = 1, 2, \ldots, N \), we intend to evaluate the optimal values \( \hat{x} \). To this end, for \( k = N \) we obtain

\[
\hat{x}_N = \arg \min_{0 \leq x_N \leq E_{\text{max}}} V_N(x_N),
\]

while for every \( k = (N - 1), \ldots, 1, \)

\[
\hat{x}_k = \arg \min_{0 \leq x_k \leq E_{\text{max}}} \left\{ V_k(x_k) + \Delta F \left( \frac{x_{k+1}}{\eta(u_{k+1})} \Delta + P_{L,k+1} \right) \right\}.
\]

Finally, the globally optimal control signal is

\[
\hat{u}_k = \begin{cases} 
\frac{x_{k+1}}{\eta(u_k)} \Delta, & \text{for } k = 1, \\
\frac{x_k}{\eta(u_k)} \Delta - \frac{x_{k-1}}{\eta(u_{k-1})} \Delta, & \text{otherwise.}
\end{cases}
\]
2) Lossy Case: We consider here the case of a lossy storage device (with $\eta = 0.8$), as detailed in Section III. The parameters are similar to the ones presented above (see Fig. 3), and are therefore omitted. It is important to stress that here we are testing the full implementation of the dynamic programming algorithm, which includes all possible cases. This makes the algorithm more computationally demanding even in the lossless case ($\eta = 1$).

Figures 7 and 8 show the RMSE measure and elapsed run-time for both methods. We use the same load as in the previous setup, and keep the set of parameters almost the same. First, for the minimum principle we change the accuracy of the initial value in the range $\lambda_0 \in [0.01, 0.5]$, while the number of time samples is fixed at $N = 70$. Next, the number of time samples varies in the range $N \in [4, 100]$ while $\lambda_0 = 0.01$. We start with a slightly larger $N$ due to convergence issues that occur with a low number of time samples. For the dynamic programming algorithm we first take $M \in [50, 1000]$ with a step of 50 while $N = 70$, and then use $N \in [5, 150]$ with a step of 5 and $M = 1200$. These steps are used due to the high computational complexity. The reference signals are separately generated using the last model for each of the methods. The results show that after the minimum principle algorithm approaches a plateau further progress in computation time and accuracy is limited. In addition, the dynamic programming algorithm shows exponential and linear trends for the energy
and time samples; however, the magnitude at the end of the simulation period becomes much higher. Hence, dynamic programming is useful with a low number of samples when high accuracy is not significant. In addition, Fig. 9 depicts the scaled levels of the elapsed time. Both methods result in similar patterns as in the lossless case.

3) Robustness analysis: In this section we aim to perform additional experiments on the “Ashalim” molten salt storage system to study the robustness of the proposed solution. To this end, we consider two scenarios: (i) the storage system is badly modeled, and (ii) the load profile $P_L(t)$ is wrongly measured. We use similar parameters as given in Section IV-B.

Table I summarizes how the two algorithms respond to modeling errors, showing that both methods are relatively accurate and have similar measures. The specified percentage represents an intentional error in efficiency, and the resulting signals are compared to the ideal shortest-path algorithm.

Next, we assume that $P_L(t)$ is wrongly measured, and has an additive noise signal. In simulation this noise signal is modeled as a sequence of uniformly distributed random numbers which range from 0 to a maximal value that is either 0.02, 0.04, or 0.05. For these values different statistical measures are computed for both the minimum principle and dynamic programming based algorithms. The results are summarized in Table II, in which we first present the relative error when $P_L(t)$ is correctly measured (denoted as “ideal”), and then present the relative error when $P_L(t)$ is corrupted with additive noise. The minimum principle appears to be slightly more robust and stable since its respective errors grow slower than those related to the dynamic programming algorithm.

<table>
<thead>
<tr>
<th>Measure</th>
<th>Method</th>
<th>1% error</th>
<th>5% error</th>
<th>8% error</th>
</tr>
</thead>
<tbody>
<tr>
<td>RMSE $(\Delta E_g)$</td>
<td>PMP</td>
<td>0.0150</td>
<td>0.0459</td>
<td>0.0737</td>
</tr>
<tr>
<td></td>
<td>DP</td>
<td>0.0119</td>
<td>0.0436</td>
<td>0.0722</td>
</tr>
<tr>
<td>SSE $(\Delta E_g)$</td>
<td>PMP</td>
<td>0.2262</td>
<td>2.1048</td>
<td>5.4381</td>
</tr>
<tr>
<td></td>
<td>DP</td>
<td>0.1427</td>
<td>1.9007</td>
<td>5.2139</td>
</tr>
<tr>
<td>ME $(\Delta E_g)$</td>
<td>PMP</td>
<td>0.0128</td>
<td>0.0367</td>
<td>0.0546</td>
</tr>
<tr>
<td></td>
<td>DP</td>
<td>-0.0045</td>
<td>0.0228</td>
<td>0.0440</td>
</tr>
<tr>
<td>IAE $(\Delta E_g)$</td>
<td>PMP</td>
<td>0.3207</td>
<td>0.8810</td>
<td>1.3106</td>
</tr>
<tr>
<td></td>
<td>DP</td>
<td>0.2590</td>
<td>0.7182</td>
<td>1.1764</td>
</tr>
<tr>
<td>max$(\Delta P_g)$</td>
<td>PMP</td>
<td>0.0549</td>
<td>0.0519</td>
<td>0.0518</td>
</tr>
<tr>
<td></td>
<td>DP</td>
<td>0.0511</td>
<td>0.0551</td>
<td>0.0597</td>
</tr>
</tbody>
</table>

Table II

<table>
<thead>
<tr>
<th>Measure</th>
<th>Method</th>
<th>Ideal</th>
<th>0.02 error</th>
<th>0.04 error</th>
<th>0.05 error</th>
</tr>
</thead>
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<tr>
<td>RMSE $(\Delta E_g)$</td>
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<td>0.0454</td>
<td>0.0973</td>
<td>0.2254</td>
<td>0.2902</td>
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<tr>
<td></td>
<td>DP</td>
<td>0.0151</td>
<td>0.1337</td>
<td>0.2666</td>
<td>0.3325</td>
</tr>
<tr>
<td>SSE $(\Delta E_g)$</td>
<td>PMP</td>
<td>0.2058</td>
<td>0.9472</td>
<td>5.0810</td>
<td>8.4212</td>
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<tr>
<td></td>
<td>DP</td>
<td>0.0228</td>
<td>1.7863</td>
<td>7.1097</td>
<td>11.0538</td>
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<tr>
<td>ME $(\Delta E_g)$</td>
<td>PMP</td>
<td>0.0374</td>
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<td>-0.1854</td>
<td>-0.2411</td>
</tr>
<tr>
<td></td>
<td>DP</td>
<td>-0.0071</td>
<td>-0.1205</td>
<td>-0.2351</td>
<td>-0.2924</td>
</tr>
<tr>
<td>IAE $(\Delta E_g)$</td>
<td>PMP</td>
<td>0.9075</td>
<td>1.8219</td>
<td>4.4507</td>
<td>5.7839</td>
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<tr>
<td></td>
<td>DP</td>
<td>0.1752</td>
<td>2.8854</td>
<td>5.6339</td>
<td>7.0082</td>
</tr>
<tr>
<td>max$(\Delta P_g)$</td>
<td>PMP</td>
<td>0.0568</td>
<td>0.0583</td>
<td>0.0678</td>
<td>0.0726</td>
</tr>
<tr>
<td></td>
<td>DP</td>
<td>0.0501</td>
<td>0.0418</td>
<td>0.0480</td>
<td>0.0578</td>
</tr>
</tbody>
</table>
C. Additional Tests with Electric Vehicles

In this section we perform a series of numerical simulations on electric vehicles to verify the applicability of the proposed control methods. We consider a family of electric vehicles which data can be found in the “Argonne National Laboratory” database [41].

We first consider the Volkswagen e-Golf electric vehicle with a 85 kW, 270 Nm permanent magnet synchronous AC motor and a 24.2 kWh, 323 V rated Lithium-ion battery. In simulations we define $E_{\text{max}} = 300$ J and select the simulation time as $T = 45$ sec, and the sample time as $\Delta T = 0.1$. For the dynamic programming algorithm the number of time and energy samples are 451 and 800 respectively, and for the minimum principle algorithm we choose $\rho = 10^3$. The simulation results are depicted in Fig. 10, in which the top plot shows the resulting optimal generated power and the load profile $P_L(t)$. The bottom plot presents errors in the generated power, where the shortest path signal is used as a reference.

Table III presents the peak power for other electric vehicles available in the database. The optimal power flows are computed using the shortest path, minimum principle, and dynamic programming algorithms. It can be seen that for all these vehicles all methods produce similar peak power. The database contains various car data samples. Therefore, the last column in the table indicates record serial number.

![Fig. 10. The upper plot shows the load profile and the optimal generated power as computed by the shortest path method (---), the minimum principle (--), and dynamic programming (--). The bottom plot depicts the relative errors in generated power, where the shortest-path method is used as a reference.](image)

Table III presents the peak power for several methods [in kW]

<table>
<thead>
<tr>
<th>Model name</th>
<th>Nominal</th>
<th>SP</th>
<th>PMP</th>
<th>DP</th>
<th>#</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercedes-Benz</td>
<td>13.1000</td>
<td>2.4811</td>
<td>2.4916</td>
<td>2.6195</td>
<td>79</td>
</tr>
<tr>
<td>Nissan Leaf SV</td>
<td>20.0757</td>
<td>3.0375</td>
<td>3.0505</td>
<td>3.7878</td>
<td>144</td>
</tr>
<tr>
<td>Mitsubishi I-MiEV</td>
<td>6.5887</td>
<td>1.1180</td>
<td>1.1213</td>
<td>1.3086</td>
<td>100</td>
</tr>
<tr>
<td>Chevrolet Spark EV</td>
<td>7.8028</td>
<td>1.9977</td>
<td>2.0050</td>
<td>2.0809</td>
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</tr>
<tr>
<td>Volkswagen e-Golf</td>
<td>38.6643</td>
<td>19.2181</td>
<td>19.2713</td>
<td>20.8087</td>
<td>207</td>
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<tr>
<td>Smart EV</td>
<td>16.1382</td>
<td>8.9181</td>
<td>8.9277</td>
<td>9.2266</td>
<td>188</td>
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<tr>
<td>BMW i3BEV</td>
<td>21.7496</td>
<td>5.6388</td>
<td>5.6646</td>
<td>5.9420</td>
<td>1</td>
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<tr>
<td>Ford Focus</td>
<td>8.8122</td>
<td>1.7193</td>
<td>1.7253</td>
<td>1.8819</td>
<td>42</td>
</tr>
<tr>
<td>Kia Soul</td>
<td>82.2908</td>
<td>30.2048</td>
<td>30.3018</td>
<td>31.9621</td>
<td>64</td>
</tr>
</tbody>
</table>

V. Conclusion

Although the minimum principle is not straightforward to apply for optimal control of storage devices, its main advantage is that it may provide simple solutions with low complexity. For that reason, in this paper we are presenting an optimal control policy for lossy storage devices that is based on the minimum principle. The optimal control problem solved in this article presents two main challenges: first, the continuous efficiency of the storage device leads to a non-affine Hamiltonian function; second, the finite storage capacity introduces a hard constraint on the state variables. To solve these problems, first we propose a modified cost function that incorporates this hard constraint, and then compute necessary conditions for optimality, based on the modified Hamiltonian. The proposed solution does not impose any restrictions on the device efficiency, and therefore can be used to find solutions to general efficiency functions. Furthermore, we provide a common basis for comparing the proposed solution to a dynamic programming algorithm, in order to understand which algorithm is preferable in each specific scenario. Both algorithms are compared based on the Ashalim molten-salt storage system. To make a significant comparison we use a fixed range of parameters and consider several statistical measures to evaluate the optimal control signals. In addition, we compare the computational time of each control strategy by varying the number of time samples, number of energy samples, or $\lambda_0$. The numeric experiments reveal that for lossless storage devices dynamic programming is beneficial for obtaining a fast and accurate solution when a low number of samples is used. However, for lossy storage devices the situation is the opposite, and the minimum principle provides faster and more accurate solutions, since its computational complexity is only minimally affected by changes in the system parameters.

**References**


