On Controllability of Switched Linear Systems on Time Scales

Juri Belikov, Aleksei Tepljakov

Tallinn University of Technology

jbelikov@cc.ioc.ee, aleksei.tepljakov@ttu.ee

July 16, 2015
Time scale is a model of time

**Definition 1**

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers.

- $\mathbb{T} = \mathbb{R}$: continuous time
- $\mathbb{T} = \mathbb{Z}$: discrete time
- $\mathbb{T} = \tau \mathbb{Z} := \{\tau k \mid k \in \mathbb{Z}\}$, $\tau > 0$: discrete time
- $\mathbb{T} = \mathbb{P}_{a,b} := \bigcup_{k=0}^{\infty} [k(a + b), k(a + b) + a]$: nonuniform discrete time
- Hybrid time
The forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ is defined by

$$\sigma(t) := \inf \{\tau \in \mathbb{T} \mid \tau > t\}.$$ 

The backward jump operator $\rho : \mathbb{T} \rightarrow \mathbb{T}$ is defined by

$$\rho(t) := \inf \{\tau \in \mathbb{T} \mid \tau > t\}.$$ 

The graininess function $\mu : \mathbb{T} \rightarrow [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$ 

A time scale $\mathbb{T}$ is called **homogeneous** if $\mu \equiv \text{const}$. 
The following table presents operators for typical cases of $\mathbb{T}$, where $\text{id}$ means identity operator and $\xi^\sigma = \xi \circ \sigma$.

**Table:** Basic types of operators

<table>
<thead>
<tr>
<th>$\mathbb{T}$</th>
<th>$\sigma(t)$</th>
<th>$\rho(t)$</th>
<th>$\mu(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{R}$</td>
<td>$\text{id}$</td>
<td>$\text{id}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$h\mathbb{Z}$</td>
<td>$t + h$</td>
<td>$t - h$</td>
<td>$h$</td>
</tr>
<tr>
<td>$q\mathbb{Z}$</td>
<td>$qt$</td>
<td>$t/q$</td>
<td>$(q - 1)t$</td>
</tr>
<tr>
<td>$\mathbb{P}_{a,b}$</td>
<td>$t$</td>
<td>$t$</td>
<td>$0$</td>
</tr>
<tr>
<td>$t + b$</td>
<td>$t - b$</td>
<td>$b$</td>
<td>$t \in \bigcup_{k=0}^{\infty} [k(a + b), k(a + b) + a]$</td>
</tr>
</tbody>
</table>

$t \in \mathbb{R}$
Delta derivative of $f : \mathbb{T} \to \mathbb{R}$, denoted by $f^\Delta(t)$, can be defined as the extension of standard time-derivative in the continuous-time case.

Indefinite integral:

$$\int f(t)\Delta t = F(t) + C.$$  

The Cauchy integral:

$$\int_a^b f(t)\Delta t = F(a) - F(b) \quad \text{for all} \quad a, b \in \mathbb{T}.$$
The following table presents the operators for typical cases of $\mathbb{T}$, where $a < b$.

<table>
<thead>
<tr>
<th>$\mathbb{T}$</th>
<th>$\xi^\Delta(t)$</th>
<th>$\int_a^b f(t)\Delta t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{R}$</td>
<td>$\frac{d\xi(t)}{dt}$</td>
<td>$\int_a^b f(t)dt$</td>
</tr>
<tr>
<td>$h\mathbb{Z}$</td>
<td>$\frac{\xi(t+h)-\xi(t)}{h}$</td>
<td>$\sum_{k=a/h}^{b/h-1} hf(kh)$</td>
</tr>
<tr>
<td>$q\mathbb{Z}$</td>
<td>$\frac{\xi(qt)-\xi(t)}{(q-1)t}$</td>
<td>$\sum_{t=a}^{b/q} \frac{\xi(t)}{(q-1)t}$</td>
</tr>
</tbody>
</table>
Consider a switched linear control system defined on a time scale $\mathbb{T}$

$$x^\Delta(t) = A_{s(t)}x(t) + B_{s(t)}u(t), \quad (1)$$

- $x(t) \in \mathbb{R}^n$ is a vector of state variables;
- $u(t) \in \mathbb{R}^m$ is a vector of input functions;
- $s : \mathbb{T} \rightarrow \mathcal{N} = \{1, 2, \ldots, l\}$ is the switching law to be designed;
- $A_{s(t)} \in \mathbb{R}^{n \times n}, B_{s(t)} \in \mathbb{R}^{n \times m}$ are constant matrices;
- $t \in \mathbb{T}$.
Switched Linear Control Systems

General solution

Suppose that the switching signal is well-defined and its switching sequence is \{x_0, (t_0, i_1), (t_1, i_2), \ldots, (t_{l-1}, i_l)\}. As the \(i_1\)th subsystem is active during \([t_0, t_1)\) we have

\[
x^\Delta(t) = A_{i_1} x(t) + B_{i_1} u(t)
\]

with \(x(t_0) = x_0, t \in [t_0, t_1)\). \hspace{1cm} (2)

Equation (2) is a linear delta differential equation with an initial condition, whose solution is

\[
x_1 = e^{A_{i_1} (t_1, t_0)} x_0 + \int_{t_0}^{t_1} e^{A_{i_1} (t_1, \sigma(\tau))} B_{i_1} u(\tau) \Delta \tau.
\]
Suppose that the switching signal is well-defined and its switching sequence is \( \{x_0, (t_0, i_1), (t_1, i_2), \ldots, (t_{l-1}, i_l)\} \). As the \( i_1 \)th subsystem is active during \([t_0, t_1)\) we have

\[
x^\Delta(t) = A_{i_1} x(t) + B_{i_1} u(t) \quad \text{with} \quad x(t_0) = x_0, \quad t \in [t_0, t_1). 
\]

Equation (2) is a linear delta differential equation with an initial condition, whose solution is

\[
x_1 = e^{A_{i_1}(t_1, t_0)} x_0 + \int_{t_0}^{t_1} e^{A_{i_1}(t_1, \sigma(\tau))} B_{i_1} u(\tau) \Delta \tau.
\]
Proceeding in the same manner the general solution of (1) is

\[ x(t) = e^{A_{i_k}(t, t_{k-1})} \cdots e^{A_{i_1}(t_1, t_0)} x_0 + \]

\[ e^{A_{i_k}(t, t_{k-1})} \cdots e^{A_{i_2}(t_2, t_1)} \int_{t_0}^{t_1} e^{A_{i_1}(t_1, \sigma(\tau))} B_{i_1} u(\tau) d\tau \]

\[ + \cdots + e^{A_{i_k}(t, t_{k-1})} \int_{t_{k-2}}^{t_{k-1}} e^{A_{i_{k-1}}(t_k, \sigma(\tau))} B_{i_{k-1}} u(\tau) d\tau + \]

\[ \int_{t_{k-1}}^{t} e^{A_{i_k}(t, \sigma(\tau))} B_{i_k} u(\tau) d\tau, \]

where \( t \in [t_{k-1}, t_k) \) for \( k = 1, \ldots, l \) with \( t_l = t_f \) and \( i_1 = s(t_0), \ldots, i_l = s(t_{l-1}) \).
Definition 2

System (1) is said to be **controllable** at $t_0$, if for any initial state $x_0$ and any final state $x_f$ there exist a time $t_f > t_0$, a switching path $s : [t_0, t_f] \rightarrow \mathcal{N}$, and inputs $u : [t_0, t_f] \rightarrow \mathbb{R}^p$ such that $x(t; t_0, x_0, u, s) = x_f$.

Definition 3

The **controllable set** of system (1) is the set of states which are controllable.

Definition 4

System (1) is said to be (completely) **controllable**, if its controllable set is $\mathbb{R}^n$. 
Definition 2

System (1) is said to be **controllable** at $t_0$, if for any initial state $x_0$ and any final state $x_f$ there exist a time $t_f > t_0$, a switching path $s : [t_0, t_f] \rightarrow \mathcal{N}$, and inputs $u : [t_0, t_f] \rightarrow \mathbb{R}^p$ such that $x(t; t_0, x_0, u, s) = x_f$.

Definition 3

The **controllable set** of system (1) is the set of states which are controllable.

Definition 4

System (1) is said to be (completely) controllable, if its controllable set is $\mathbb{R}^n$. 
Definition 2
System (1) is said to be **controllable** at $t_0$, if for any initial state $x_0$ and any final state $x_f$ there exist a time $t_f > t_0$, a switching path $s : [t_0, t_f] \to \mathcal{N}$, and inputs $u : [t_0, t_f] \to \mathbb{R}^p$ such that $x(t; t_0, x_0, u, s) = x_f$.

Definition 3
The **controllable set** of system (1) is the set of states which are controllable.

Definition 4
System (1) is said to be (completely) controllable, if its controllable set is $\mathbb{R}^n$. 
Let us define the **controllability matrix** of the pair $(A_{ik}, B_{ik})$, for $k = 1, \ldots, l$, as

$$
C_{ik} := \begin{bmatrix} B_{ik} & A_{ik} B_{ik} & \cdots & A_{ik}^{n-1} B_{ik} \end{bmatrix}.
$$

Then, the collection of matrices $C_{ik}$ can be denoted by

$$
C := \begin{bmatrix} C_{i1} & C_{i2} & \cdots & C_{il} \end{bmatrix}.
$$

**Theorem 5**

*The switched linear system (1) with $l$ modes is controllable, if the controllability matrix $C$ is of full rank, i.e., rank $C = n$.***
Let us define the **controllability matrix** of the pair \((A_{ik}, B_{ik})\), for \(k = 1, \ldots, l\), as

\[
C_{ik} := \begin{bmatrix}
B_{ik} & A_{ik} B_{ik} & \cdots & A_{ik}^{n-1} B_{ik}
\end{bmatrix}.
\]

Then, the collection of matrices \(C_{ik}\) can be denoted by

\[
C := \begin{bmatrix}
C_{i1} & C_{i2} & \cdots & C_{il}
\end{bmatrix}.
\]

**Theorem 5**

*The switched linear system (1) with \(l\) modes is controllable, if the controllability matrix \(C\) is of full rank, i.e., \(\text{rank}\ C = n\).*
Consider a binary-mode switched linear control system (1) defined on a time scale $\mathbb{T}$

$$
A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \end{bmatrix}.
$$

The controllability matrix can be calculated as $C = \begin{bmatrix} C_{i1} & C_{i2} \end{bmatrix}$, where

$$
C_{i1} = \begin{bmatrix} B_1 & A_1 B_1 & A_1 B_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
$$

$$
C_{i2} = \begin{bmatrix} B_2 & A_2 B_2 & A_2 B_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.
$$

Observe that separately the subsystems are not controllable, since

$$
\text{rank} \, C_{i1} = 2 \quad \text{and} \quad \text{rank} \, C_{i2} = 1.
$$

However, according to Theorem 5, the overall switched system is controllable, since

$$
\text{rank} \, C = 3.
$$
Consider a binary-mode switched linear control system (1) defined on a time scale $T$

\[
A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
\]

The controllability matrix can be calculated as $C = [C_{i_1} \quad C_{i_2}]$, where

\[
C_{i_1} = \begin{bmatrix} B_1 & A_1 B_1 & A_1 B_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

\[
C_{i_2} = \begin{bmatrix} B_2 & A_2 B_2 & A_2 B_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.
\]

Observe that separately the subsystems are not controllable, since

$\text{rank } C_{i_1} = 2$ and $\text{rank } C_{i_2} = 1$.

However, according to Theorem 5, the overall switched system is controllable, since

$\text{rank } C = 3$. 

Consider a binary-mode switched linear control system (1) defined on a time scale $\mathbb{T}$

$$
A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
$$

The controllability matrix can be calculated as $C = [C_{i_1} \quad C_{i_2}]$, where

$$
C_{i_1} = \begin{bmatrix} B_1 & A_1B_1 & A_1B_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
$$

$$
C_{i_2} = \begin{bmatrix} B_2 & A_2B_2 & A_2B_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.
$$

Observe that separately the subsystems are not controllable, since

$$
\text{rank} \ C_{i_1} = 2 \text{ and } \text{rank} \ C_{i_2} = 1.
$$

However, according to Theorem 5, the overall switched system is controllable, since

$$
\text{rank} \ C = 3.
$$
Controllability of switched systems

Necessity

Let

$$\gamma^k(i_1, \ldots, i_k) = \begin{bmatrix} A_{i_k}^{j_k} \cdots A_{i_1}^{j_1} B_{i_1} \end{bmatrix}_{j_k, \ldots, j_1 \in \{0,1,\ldots,n-1\}},$$

where power indices $j_1, \ldots, j_k$ take all the possible values from the set $\{0, 1 \ldots, n - 1\}$ and mode indices $i_1, \ldots, i_k$ take all possible values from the set $\mathcal{N}$. Next, based on the definition of $\gamma^k$ a new matrix $\Gamma^k$ can be constructed as

$$\Gamma^0(i) = \gamma^1(i), \ldots, \Gamma^k(i) = \begin{bmatrix} \gamma^{k+1}(i, i_1, \ldots, i_k) \end{bmatrix}_{i_1, \ldots, i_k \in \mathcal{N}},$$

with $i_1 \neq i, \ldots, i_k \neq i_{k-1}$. Now, the system joint controllability matrices are defined as

$$\mathcal{W}^0 = \begin{bmatrix} \Gamma^0(1) & \cdots & \Gamma^0(l) \end{bmatrix}, \ldots, \mathcal{W}^k = \begin{bmatrix} \Gamma^k(1) & \cdots & \Gamma^k(l) \end{bmatrix}.$$

$\mathcal{W}^k$ is the $k$th-order joint controllability matrix.

Theorem 6

If system (1) is controllable, then the $k$th-order system joint controllability matrix is of full rank, i.e., $\text{rank} \mathcal{W}^k = n$. 
Let
\[ \gamma^k(i_1, \ldots, i_k) = \begin{bmatrix} A_{i_k}^{j_k} \cdots A_{i_1}^{j_1} B_{i_1} \end{bmatrix}_{j_k, \ldots, j_1 \in \{0, 1, \ldots, n-1\}} \]
where power indices \( j_1, \ldots, j_k \) take all the possible values from the set \( \{0, 1, \ldots, n-1\} \) and mode indices \( i_1, \ldots, i_k \) take all possible values from the set \( \mathcal{N} \). Next, based on the definition of \( \gamma^k \) a new matrix \( \Gamma^k \) can be constructed as
\[ \Gamma^0(i) = \gamma^1(i), \ldots, \Gamma^k(i) = \begin{bmatrix} \gamma^{k+1}(i, i_1, \ldots, i_k) \end{bmatrix}_{i_1, \ldots, i_k \in \mathcal{N}} \]
with \( i_1 \neq i, \ldots, i_k \neq i_{k-1} \). Now, the system joint controllability matrices are defined as
\[ \mathcal{W}^0 = \begin{bmatrix} \Gamma^0(1) & \cdots & \Gamma^0(l) \end{bmatrix}, \ldots, \mathcal{W}^k = \begin{bmatrix} \Gamma^k(1) & \cdots & \Gamma^k(l) \end{bmatrix}. \]
\( \mathcal{W}^k \) is the \( k \)-th-order joint controllability matrix.

**Theorem 6**

*If system (1) is controllable, then the \( k \)-th-order system joint controllability matrix is of full rank, i.e., \( \text{rank} \mathcal{W}^k = n \).*
Consider a switched linear control system with two subsystems

\[ A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \end{bmatrix}. \]

The joint controllability matrix is

\[ W^2 = \begin{bmatrix} \Gamma^2(1) & \Gamma^2(2) \end{bmatrix} = \begin{bmatrix} B_1 & A_1 & A_1 B_1 & A_2 A_1 B_1 & A_2 B_1 & B_2 & A_2 & A_2 B_2 & A_1 A_2 B_2 & A_1 B_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \]

One may easily check that \( \text{rank} \ W^2 = 1. \) Hence, the necessary condition of Theorem 6 is not satisfied, and therefore, the system is not controllable.
Conclusions

- The paper addresses the controllability problem of switched linear systems defined on a time scale.

- The algebraic point of view is used to derive unified conditions.

- The time scales calculus based approach is used to justify the validity of obtained conditions for a large class of systems defined in different time domains accommodated by recalled formalism.

- The necessary and sufficient controllability conditions are separately formulated in terms of specific matrices that are extensions of those derived for LTI systems.
Controllability of switched systems
Motivating example for further research

Consider a model of PWM-driven boost converter in the form of switched linear control system (1) with two subsystems defined on a time scale $\mathbb{T}$

$$A_1 = \begin{bmatrix} -\frac{1}{RC} & 1 \\ \frac{1}{C} & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -\frac{1}{RC} & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix},$$

where the constant parameters $L$, $C$, $R$ represent, respectively, the inductance, capacitance, load resistance. Note that $s(t) \in \{1, 2\}$ and $n = 2$, $m = 1$, $l = 2$.

**Sufficiency:** the controllability matrix can be calculated as $C = \left[ C_i^1 \quad C_i^2 \right]$, where

$$C_i^1 = \begin{bmatrix} B_1 \quad A_1 B_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_i^2 = \begin{bmatrix} B_2 \quad A_2 B_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{1}{L} & 0 \end{bmatrix}.$$

The sufficient condition given in Theorem 5 is not satisfied, since $\text{rank} \ C = 1$.

**Necessity:** the joint controllability matrix is

$$\mathcal{W}^2 = \left[ \Gamma^2(1) \quad \Gamma^2(2) \right] = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{CL} & 0 & 0 & 0 \\ \frac{1}{L} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The necessary controllability condition of Theorem 6 is satisfied, since $\text{rank} \ \mathcal{W}^2 = 2$. 
Consider a model of PWM-driven boost converter in the form of switched linear control system (1) with two subsystems defined on a time scale $T$

$$A_1 = \begin{bmatrix} -\frac{1}{RC} & 1 \\ \frac{1}{C} & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -\frac{1}{RC} & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix},$$

where the constant parameters $L, C, R$ represent, respectively, the inductance, capacitance, load resistance. Note that $s(t) \in \{1, 2\}$ and $n = 2, m = 1, l = 2$.

**Sufficiency:** the controllability matrix can be calculated as $C = [C_{i_1}, C_{i_2}]$, where

$$C_{i_1} = [B_1, A_1 B_1] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_{i_2} = [B_2, A_2 B_2] = \begin{bmatrix} 0 & 0 \\ \frac{1}{L} & 0 \end{bmatrix}.$$

The sufficient condition given in Theorem 5 is not satisfied, since $\text{rank } C = 1$.

**Necessity:** the joint controllability matrix is

$$\mathcal{W}^2 = [\Gamma^2(1), \Gamma^2(2)] = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{CL} & 0 & 0 & 0 \\ \frac{1}{L} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The necessary controllability condition of Theorem 6 is satisfied, since $\text{rank } \mathcal{W}^2 = 2$. 
Consider a model of PWM-driven boost converter in the form of switched linear control system (1) with two subsystems defined on a time scale $\mathbb{T}$

\[
A_1 = \begin{bmatrix} -\frac{1}{RC} & 1 \\ \frac{1}{L} & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -\frac{1}{RC} & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix},
\]

where the constant parameters $L$, $C$, $R$ represent, respectively, the inductance, capacitance, load resistance. Note that $s(t) \in \{1, 2\}$ and $n = 2$, $m = 1$, $l = 2$.

**Sufficiency:** the controllability matrix can be calculated as $C = [C_i_1 \quad C_i_2]$, where

\[
C_i_1 = [B_1 \quad A_1 B_1] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_i_2 = [B_2 \quad A_2 B_2] = \begin{bmatrix} 0 & 0 \\ \frac{1}{L} & 0 \end{bmatrix}.
\]

The sufficient condition given in Theorem 5 is not satisfied, since $\text{rank} \, C = 1$.

**Necessity:** the joint controllability matrix is

\[
W^2 = [\Gamma^2(1) \quad \Gamma^2(2)] = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{CL} & 0 & 0 & 0 \\ \frac{1}{L} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
\]

The necessary controllability condition of Theorem 6 is satisfied, since $\text{rank} \, W^2 = 2$. 

\[2\]
Thank you very much for your attention!

Any questions?