On Controllability of Switched Linear Systems on Time Scales

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Time scale is a model of time

Definition 1

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers.

$\mathbb{T} = \mathbb{R}$ continuous time

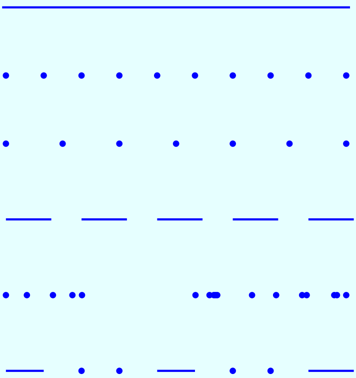
$\mathbb{T} = \mathbb{Z}$ discrete time

$\mathbb{T} = \tau\mathbb{Z} := \{\tau k \mid k \in \mathbb{Z}\}, \ \tau > 0$ discrete time

$$\mathbb{T} = \mathbb{P}_{a,b} := \bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+a]$$

Nonuniform discrete time

Hybrid time



- The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf \{\tau \in \mathbb{T} \mid \tau > t\}.$$

- The backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \inf\{\tau \in \mathbb{T} \mid \tau > t\}.$$

- The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

A time scale $\mathbb{T}$ is called homogeneous if $\mu \equiv \mathrm{const}.$

The following table presents operators for typical cases of $\mathbb{T}$, where id means identity operator and $\xi^\sigma = \xi \circ \sigma$.

Table: Basic types of operators

$\mathbb{T}$	$\sigma(t)$	$\rho(t)$	$\mu(t)$	
$\mathbb{R}$	id	id	0	
$h\mathbb{Z}$	$t+h$	$t-h$	h	
$\overline{q^{\mathbb{Z}}}$	qt	t/q	$(q-1)t$	
$\mathbb{P}_{a,b}$	t	t	0	$t \in \bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+a]$
	$t+b$	$t-b$	b	$t \in \bigcup_{k=0}^{\infty} \{k(a+b)+a\}$

Delta derivative of $f : \mathbb{T} \to \mathbb{R}$, denoted by $f^{\Delta}(t)$, can be defined as the extension of standard time-derivative in the continuous-time case.

Indefinite integral:

$$\int f(t)\Delta t = F(t) + C.$$

The Cauchy integral:

$$\int_a^b f(t)\Delta t = F(a) - F(b) \quad \text{for all} \quad a, b \in \mathbb{T}.$$

The following table presents the operators for typical cases of $\mathbb{T}$, where $a < b$.

Table: Delta (anti)derivative

$\mathbb{T}$	$\xi^{\Delta}(t)$	$\int_a^b f(t)\Delta t$
$\mathbb{R}$	$\frac{\mathrm{d}\xi(t)}{\mathrm{d}t}$	$\int_a^b f(t)\mathrm{d}t$
$h\mathbb{Z}$	$\frac{\xi(t+h)-\xi(t)}{h}$	$\sum_{k=a/h}^{b/h-1} hf(kh)$
$\overline{q^{\mathbb{Z}}}$	$\frac{\xi(qt)-\xi(t)}{(q-1)t}$	$\sum_{t=a}^{b/q} \frac{\xi(t)}{(q-1)t}$

Consider a switched linear control system defined on a time scale $\mathbb{T}$

$$x^{\Delta}(t) = A_{s(t)}x(t) + B_{s(t)}u(t), \tag{1}$$

- $x(t) \in \mathbb{R}^n$ is a vector of state variables;
- $u(t) \in \mathbb{R}^m$ is a vector of input functions;
- $s : \mathbb{T} \to \mathcal{N} = \{1, 2, \ldots, l\}$ is the switching law to be designed;
- $A_{s(t)} \in \mathbb{R}^{n\times n}, B_{s(t)} \in \mathbb{R}^{n\times m}$ are constant matrices;
- $t \in \mathbb{T}$.

Suppose that the switching signal is well-defined and its switching sequence is $\{x_0, (t_0, i_1), (t_1, i_2), \ldots, (t_{l-1}, i_l)\}$. As the i_1th subsystem is active during $[t_0, t_1)$ we have

$$x^{\Delta}(t) = A_{i_1} x(t) + B_{i_1} u(t) \qquad \text{with } x(t_0) = x_0, t \in [t_0, t_1). \tag{2}$$

Equation (2) is a linear delta differential equation with an initial condition, whose solution is

$$x_1 = \mathrm{e}_{A_{i_1}}(t_1, t_0) x_0 + \int_{t_0}^{t_1} \mathrm{e}_{A_{i_1}}(t_1, \sigma(\tau)) B_{i_1} u(\tau) \Delta\tau.$$

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Proceeding in the same manner the general solution of (1) is

$$x(t) = \mathrm{e}_{A_{i_k}}(t, t_{k-1}) \cdots \mathrm{e}_{A_{i_1}}(t_1, t_0)x_0 + \mathrm{e}_{A_{i_k}}(t, t_{k-1}) \cdots \mathrm{e}_{A_{i_2}}(t_2, t_1) \int_{t_0}^{t_1} \mathrm{e}_{A_{i_1}}(t_1, \sigma(\tau))B_{i_1} u(\tau)\Delta\tau + \cdots + \mathrm{e}_{A_{i_k}}(t, t_{k-1}) \int_{t_{k-2}}^{t_{k-1}} \mathrm{e}_{A_{i_{k-1}}}(t_k, \sigma(\tau))B_{i_{k-1}} u(\tau)\Delta\tau + \int_{t_{k-1}}^{t} \mathrm{e}_{A_{i_k}}(t, \sigma(\tau))B_{i_k} u(\tau)\Delta\tau,$$

where $t \in [t_{k-1}, t_k)$ for $k = 1, \ldots, l$ with $t_l =: t_f$ and $i_1 = s(t_0), \ldots, i_l = s(t_{l-1})$.

Definition 2

System (1) is said to be controllable at t_0, if for any initial state x_0 and any final state x_f there exist a time $t_f > t_0$, a switching path $s : [t_0, t_f] \to \mathcal{N}$, and inputs $u : [t_0, t_f] \to \mathbb{R}^p$ such that $x(t; t_0, x_0, u, s) = x_f$.

Definition 3

The controllable set of system (1) is the set of states which are controllable.

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System (1) is said to be (completely) controllable, if its controllable set is $\mathbb{R}^n$.

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Let us define the controllability matrix of the pair (A_{i_k}, B_{i_k}), for $k = 1, \ldots, l$, as

$$\mathcal{C}_{i_k} := \begin{bmatrix} B_{i_k} & A_{i_k} B_{i_k} & \cdots & A_{i_k}^{n-1} B_{i_k} \end{bmatrix}.$$

Then, the collection of matrices $\mathcal{C}_{i_k}$ can be denoted by

$$\mathcal{C} := \begin{bmatrix} \mathcal{C}_{i_1} & \mathcal{C}_{i_2} & \cdots & \mathcal{C}_{i_l} \end{bmatrix}.$$

Theorem 5

The switched linear system (1) *with l modes is controllable, if the controllability matrix $\mathcal{C}$ is of full rank, i.e.,* $\operatorname{rank} \mathcal{C} = n$.

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Consider a binary-mode switched linear control system (1) defined on a time scale $\mathbb{T}$

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The controllability matrix can be calculated as $\mathcal{C} = \begin{bmatrix} \mathcal{C}_{i_1} & \mathcal{C}_{i_2} \end{bmatrix}$, where

$$\mathcal{C}_{i_1} = \begin{bmatrix} B_1 & A_1 B_1 & A_1 B_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{C}_{i_2} = \begin{bmatrix} B_2 & A_2 B_2 & A_2 B_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Observe that separately the subsystems are not controllable, since

$$\operatorname{rank} \mathcal{C}_{i_1} = 2 \text{ and } \operatorname{rank} \mathcal{C}_{i_2} = 1.$$

However, according to Theorem 5, the overall switched system is controllable, since

$$\operatorname{rank} \mathcal{C} = 3.$$

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Let

$$\gamma^k(i_1, \dots, i_k) = \left[A_{i_k}^{j_k} \cdots A_{i_1}^{j_1} B_{i_1}\right]_{j_k, \dots, j_1 \in \{0,1,\dots,n-1\}},$$

where power indices $j_1, \dots, j_k$ take all the possible values from the set $\{0, 1 \dots, n-1\}$ and mode indices $i_1, \dots, i_k$ take all possible values from the set $\mathcal{N}$. Next, based on the definition of γ^k a new matrix Γ^k can be constructed as

$$\Gamma^0(i) = \gamma^1(i), \dots, \Gamma^k(i) = \left[\gamma^{k+1}(i, i_1, \dots, i_k)\right]_{i_1, \dots, i_k \in \mathcal{N}}$$

with $i_1 \neq i, \dots, i_k \neq i_{k-1}$. Now, the system joint controllability matrices are defined as

$$\mathcal{W}^0 = \begin{bmatrix}\Gamma^0(1) & \cdots & \Gamma^0(l)\end{bmatrix}, \dots, \mathcal{W}^k = \begin{bmatrix}\Gamma^k(1) & \cdots & \Gamma^k(l)\end{bmatrix}.$$

$\mathcal{W}^k$ is the kth-order joint controllability matrix.

Theorem 6

If system (1) is controllable, then the kth-order system joint controllability matrix is of full rank, i.e., $\operatorname{rank} \mathcal{W}^k = n$.

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Consider a switched linear control system with two subsystems

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The joint controllability matrix is

$$\begin{aligned} \mathcal{W}^2 &= \begin{bmatrix} \Gamma^2(1) & \Gamma^2(2) \end{bmatrix} \\ &= \begin{bmatrix} B_1 & A_1 & A_1 B_1 & A_2 A_1 B_1 & A_2 B_1 & B_2 & A_2 & A_2 B_2 & A_1 A_2 B_2 & A_1 B_2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

One may easily check that $\operatorname{rank} \mathcal{W}^2 = 1$. Hence, the necessary condition of Theorem 6 is not satisfied, and therefore, the system is not controllable.

Conclusions

- The paper addresses the controllability problem of switched linear systems defined on a time scale
- The algebraic point of view is used to derive unified conditions.
- The time scales calculus based approach is used to justify the validity of obtained conditions for a large class of systems defined in different time domains accommodated by recalled formalism.
- The necessary and sufficient controllability conditions are separately formulated in terms of specific matrices that are extensions of those derived for LTI systems.

Consider a model of PWM-driven boost converter in the form of switched linear control system (1) with two subsystems defined on a time scale $\mathbb{T}$

$$A_1 = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} -\frac{1}{RC} & 0 \\ 0 & 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix},$$

where the constant parameters L, C, R represent, respectively, the inductance, capacitance, load resistance. Note that $s(t) \in \{1, 2\}$ and $n = 2$, $m = 1$, $l = 2$.
Sufficiency: the controllability matrix can be calculated as $\mathcal{C} = \begin{bmatrix} \mathcal{C}_{i_1} & \mathcal{C}_{i_2} \end{bmatrix}$, where

$$\mathcal{C}_{i_1} = \begin{bmatrix} B_1 & A_1 B_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{C}_{i_2} = \begin{bmatrix} B_2 & A_2 B_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{1}{L} & 0 \end{bmatrix}.$$

The sufficient condition given in Theorem 5 is not satisfied, since $\operatorname{rank} \mathcal{C} = 1$.
Necessity: the joint controllability matrix is

$$\mathcal{W}^2 = \begin{bmatrix} \Gamma^2(1) & \Gamma^2(2) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{CL} & 0 & 0 & 0 \\ \frac{1}{L} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The necessary controllability condition of Theorem 6 is satisfied, since $\operatorname{rank} \mathcal{W}^2 = 2$.

Consider a model of PWM-driven boost converter in the form of switched linear control system (1) with two subsystems defined on a time scale $\mathbb{T}$

$$A_1 = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} -\frac{1}{RC} & 0 \\ 0 & 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix},$$

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The necessary controllability condition of Theorem 6 is satisfied, since $\operatorname{rank} \mathcal{W}^2 = 2$.

Thank you very much for your attention!

Any questions?