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Robust FOPI and FOPID Controller Design for FFOPDT Plants in Embedded Control Applications using Frequency-domain Analysis

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Motivation, contribution, and outline

- PID controllers are widely used for industrial process control due to their relative simplicity and applicability to a wide range of industrial control problems. Fractional-order PID controllers offer more tuning flexibility;
- The main contributions of the presented paper:
  - We derive the frequency-domain characteristics necessary for robust FOPI or FOPID controller design based on constrained optimization;
  - Since the equations cannot usually be solved algebraically, a modified Newton-Raphson method is proposed. It is tailored to each problem of finding a particular crossover frequency;
  - Having such a set of equations and means of solving them makes it possible for the control engineer to select a particular optimization algorithm based on frequency-domain evaluation of performance criteria;
- Finally, conclusions and further research perspectives are given.

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Basics of fractional calculus

Fractional calculus is a generalization of integration and differentiation to non-integer order operator ${}_a\mathscr{D}_t^\alpha$, where a and t are the limits of the operation and $\alpha \in \mathbb{R}$ is the fractional order

$$ {}_a\mathscr{D}_t^\alpha = \begin{cases} \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} & \Re(\alpha) > 0, \\ 1 & \Re(\alpha) = 0, \\ \int_a^t (\mathrm{d}\tau)^{-\alpha} & \Re(\alpha) < 0. \end{cases} \quad (1) $$

The following relation holds for noninteger exponentiation of the imaginary unit j and is frequently encountered in fractional calculus. We shall make extensive use of it throughout this talk.

$$ j^\alpha = \cos\left(\frac{\alpha\pi}{2}\right) + j\sin\left(\frac{\alpha\pi}{2}\right) \quad (2) $$

Approximation of fractional operators: The Oustaloup filter

The Oustaloup recursive filter gives a very good approximation of fractional operators in a specified frequency range and is widely used in fractional calculus. For a frequency range (ω_b, ω_h) and of order N the filter for an operator $s^\gamma, 0 < \gamma < 1$, is given by

$$\begin{aligned} s^\gamma &\approx K \prod_{k=-N}^{N} \frac{s + \omega_k'}{s + \omega_k}, \quad K = \omega_h^\gamma, \quad \omega_r = \frac{\omega_h}{\omega_b}, \\ \omega_k' &= \omega_b (\omega_r)^{\frac{k+N+\frac{1}{2}(1-\gamma)}{2N+1}}, \quad \omega_k = \omega_b (\omega_r)^{\frac{k+N+\frac{1}{2}(1+\gamma)}{2N+1}}. \end{aligned} \tag{3}$$

The resulting model order is $2N + 1$.

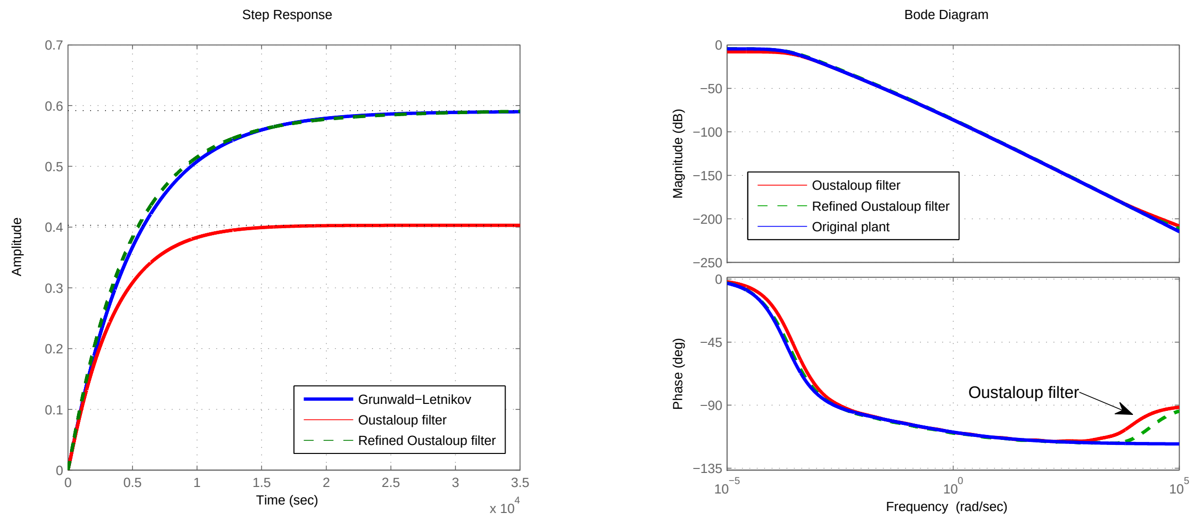
A modified Oustaloup filter has also been proposed in literature.

Oustaloup filter approximation example

Recall the fractional-order transfer function

$$G(s) = \frac{1}{14994s^{1.31} + 6009.5s^{0.97} + 1.69},$$

and approximation parameters $\omega = [10^{-4};\, 10^{4}]$, $N = 5$.



Process models

Consider the following generalizations of conventional process models used in industrial control design.

(FO)FOPDT	$G(s) = \frac{K}{1+Ts}\mathrm{e}^{-Ls}$	$G(s) = \frac{K}{1+Ts^{\alpha}}\mathrm{e}^{-Ls}$
(FO)IPDT	$G(s) = \frac{K}{s}\mathrm{e}^{-Ls}$	$G(s) = \frac{K}{s^{\alpha}}\mathrm{e}^{-Ls}$
(FO)FOIPDT	$G(s) = \frac{K}{s(1+Ts)}\mathrm{e}^{-Ls}$	$G(s) = \frac{K}{s(1+Ts^{\alpha})}\mathrm{e}^{-Ls}$

Fractional-order controllers

The fractional PI$^\lambda$D$^\mu$ controller, where λ and μ denote the orders of the integral and differential components, respectively, is given by

$$C(s) = K_p + K_i s^{-\lambda} + K_d \cdot s^{\mu}. \tag{4}$$

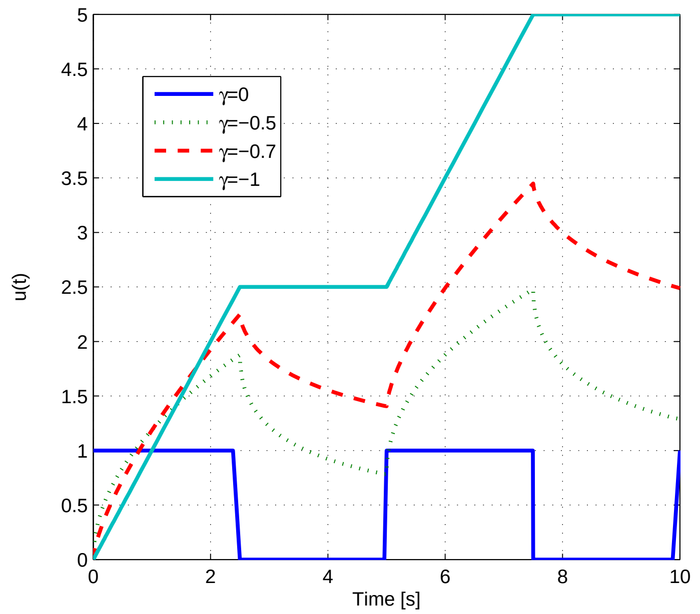
The transfer function, corresponding to the fractional lead-lag compensator of order α, has the following form:

$$C_L(s) = K \left(\frac{1 + bs}{1 + as} \right)^{\alpha}. \tag{5}$$

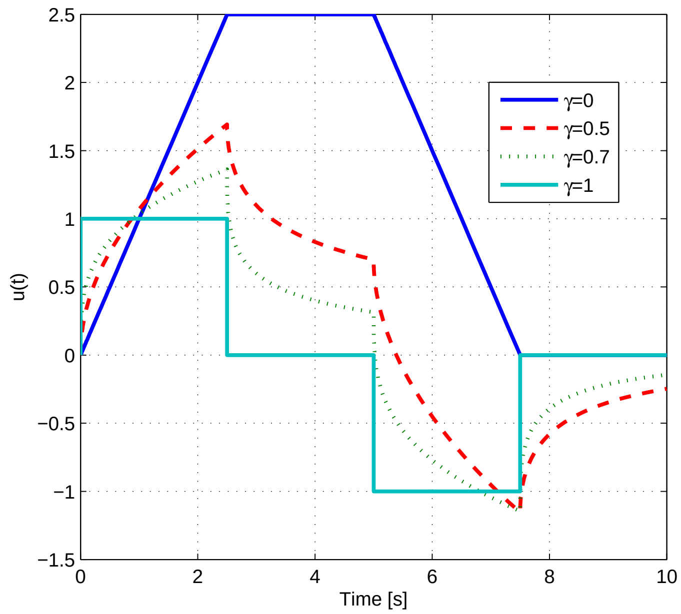
When $\alpha > 0$ we have the fractional zero and pole frequencies $\omega_z = 1/b$, $\omega_h = 1/a$ and the transfer function in (5) corresponds to a fractional lead compensator. For $\alpha < 0$, a fractional lag compensator is obtained.

Basics of fractional control: fractional control actions

Let a basic fractional control action be defined as $C(s) = K \cdot s^{\gamma}$. The control actions in the time domain for $\gamma \in [-1, 1]$ with $K = 1$ under different input signals are given below.



Fractional integrator $s^{-\gamma}$



Fractional differentiator s^{γ}

The modified Newton-Raphson method

Consider the problem of finding a root ω^* of a general nonlinear equation $f(\omega) = 0$ under the constraints $\omega > 0$ and $\omega \in (\omega_b, \omega_h)$. To tackle the problem one may employ the Newton-Raphson method which usually provides quadratic convergence to the solution. The process of locating the root starts at an initial guess ω_0 and is given by the following iterative formula:

$$\omega_{k+1} = \omega_k - f(\omega)\left(f'(\omega)\right)^{-1}. \tag{6}$$

Once a prescribed iteration limit ν is reached, or the necessary tolerance ϵ is achieved under the condition $f(\omega_k) < \epsilon$, the algorithm shall stop returning the root ω^*. However, there is a drawback of this algorithm such that local minima of $f(\omega)$ may lead to the change of sign of $f'(\omega)$ and a violation of the condition $\omega_m > 0$ at iteration step $m = k + 1$ may occur. To rectify this, the locally obtained solution at step n may be replaced such that $\omega_n = \gamma_c \omega_k$, where $\gamma_c \neq 1$ is some predefined positive factor. If as a result of this modification ω_m no longer belongs to the interval (ω_b, ω_h), the process shall fail returning $\omega^* = 0$ thereby indicating that it could not find a solution.

procedure $\textsc{Newton}(\omega_0, \gamma_c, \omega_b, \omega_h, f, f')$
  $\epsilon \leftarrow$ Tolerance, $\nu \leftarrow$ MaxIterations
  $k \leftarrow 0$; $\omega_k \leftarrow \omega_0$
  while $k < \nu$ and $f(\omega) > \epsilon$ **do**
    $\omega_{k+1} \leftarrow \omega_k - f(\omega_k)(f'(\omega_k))^{-1}$
    if $\omega_{k+1} < 0$ **then**
      $\omega_{k+1} \leftarrow \gamma_c \cdot \omega_k$
    end if
    if $\omega_{k+1} < \omega_b$ or $\omega_{k+1} > \omega_h$ **then**
      return 0
    end if
    $k \leftarrow k + 1$
  end while
  return ω_k
end procedure

The FO-FOPDT process model

Recall, that the FO-FOPDT model is given by the following transfer function

$$G(s) = \frac{K\mathrm{e}^{-Ls}}{Ts^{\alpha}+1}, \tag{7}$$

where it is assumed that $K > 0$, $T > 0$, $L > 0$ and $\alpha \in (0, 2]$. We suppose that all of the parameters of this plant are known *a priori*. They may be obtained, for instance, by employing an identification procedure of a real life process. We begin the analysis by deriving the equations to obtain the magnitude and phase angle of $G(j\omega)$. This is done by replacing $s = j\omega$ in (7), employing (2), and isolating the real and complex parts of the resulting expression as $z = a + jb$. Then, the magnitude A and phase angle φ are simply computed as $A = \sqrt{a^2 + b^2}, \varphi = \tan^{-1}(b/a)$.

The FO-FOPDT process model: Gain crossover frequency and phase margin

Next we derive open-loop characteristics of this plant. We begin by obtaining the gain crossover frequency ω_c, for which it holds

$$\left|G(j\omega_c)\right| = 1.$$

Solving this equation yields

$$\omega_c = \left(\frac{\sqrt{K^2 + \cos^2\left(\frac{\alpha\pi}{2}\right) - 1} - \cos\left(\frac{\alpha\pi}{2}\right)}{T} \right)^{1/\alpha}. \tag{8}$$

The phase margin φ_m of the system can then be determined from

$$\varphi_m = \pi - \arg\left(G(j\omega_c)\right) + 2\pi n, \quad n \geqslant 0. \tag{9}$$

The FO-FOPDT process model: Phase crossover frequency and gain margin (1)

It is more difficult to derive a formula to find the phase crossover frequency, also referred to as the ultimate frequency of the system ω_u, since we need to solve a transcendental equation

$$-L\omega_u - \tan^{-1}\left(\frac{T\sin\left(\frac{\alpha\pi}{2}\right)}{\omega_u^{-\alpha} + T\cos\left(\frac{\alpha\pi}{2}\right)}\right) = -\pi - 2\pi n, \tag{10}$$

where n is determined by the requirement to obtain a minimum gain margin $1/\left|G(j\omega_u)\right|$ closest to unity. While ω_u is usually obtained during relay autotuning, if it is not given, then the following method may be used to compute it from the FFOPDT model parameters. We first introduce a function

$$\upsilon(\omega) = \arg\left(G(j\omega)\right) + \pi + 2\pi n. \tag{11}$$

The FO-FOPDT process model: Phase crossover frequency and gain margin (2)

We compute the derivative $\mathrm{d}\upsilon(\omega)/\mathrm{d}\omega$. After simplification we arrive at

$$\upsilon'(\omega) = -L - \frac{\alpha T \sin\left(\frac{\alpha\pi}{2}\right)}{\omega\left(2T\cos\left(\frac{\alpha\pi}{2}\right) + \omega^{-\alpha} + T^2\omega^{\alpha}\right)}. \tag{12}$$

We may now use the modified Newton's method to obtain ω_u. Note, that to locate the minimum stability margin we need to introduce a modification to the search algorithm, whereby instead of terminating upon obtaining a solution ω_u^* the gain margin $1/\left|G(j\omega)\right|$ at this frequency is checked. If it is found to be less than unity, the iterative process is repeated assigning $\omega_g \leftarrow \omega_u^*$, $\omega_0 \leftarrow \omega_u^*$ and $n \leftarrow n+1$. This means that the search direction must be positive. The gain margin are then determined by means of

$$K_c = \min\left(\left|1 - 1/G(j\omega_g)\right|, \left|1 - 1/G(j\omega_u)\right|\right). \tag{13}$$

Note, that the search interval $\omega \in (\omega_b, \omega_h)$ is related to the band-limited Oustaloup approximation of a suitable fractional-order controller.

FO-FOPDT plant and FOPID controller: Open-loop characteristics

We can now derive the equations to compute the critical frequencies and corresponding stability margins of the open-loop control system given by $G_{ol}(j\omega) = C(j\omega)G(j\omega)$. A function $\psi_{pm}(\omega)$ for the phase margin is defined as

$$\psi_{pm}(\omega) := \left|C(j\omega)\right| \cdot \left|G(j\omega)\right| - 1 \tag{14}$$

To use the modified Newton-Raphson method to solve for ω_c the equation $\psi_{pm}(\omega_c) = 0$ we need to compute the derivative $\psi'_{pm}(\omega)$. In the same manner, define the function $\psi_{gm}(\omega)$ for the gain margin

$$\psi_{gm}(\omega) = \arg\left(C(j\omega)\right) + \arg\left(G(j\omega)\right) + \pi + 2\pi n \tag{15}$$

and take the derivative $\psi'_{gm}(\omega)$. Then solve $\psi_{gm}(\omega_u) = 0$ for ω_u. Finally, to check whether phase is flat at ω_c, ensuring the robustness to gain variations specification, one may check whether the following relation holds

$$\psi'_{gm}(\omega_c) = 0. \tag{16}$$

The equations to compute the sensitivity functions are derived in the same way.

Example: Computing the Gain Crossover Frequency

Suppose that a control system is given comprising a FFOPDT plant in (7) with parameters

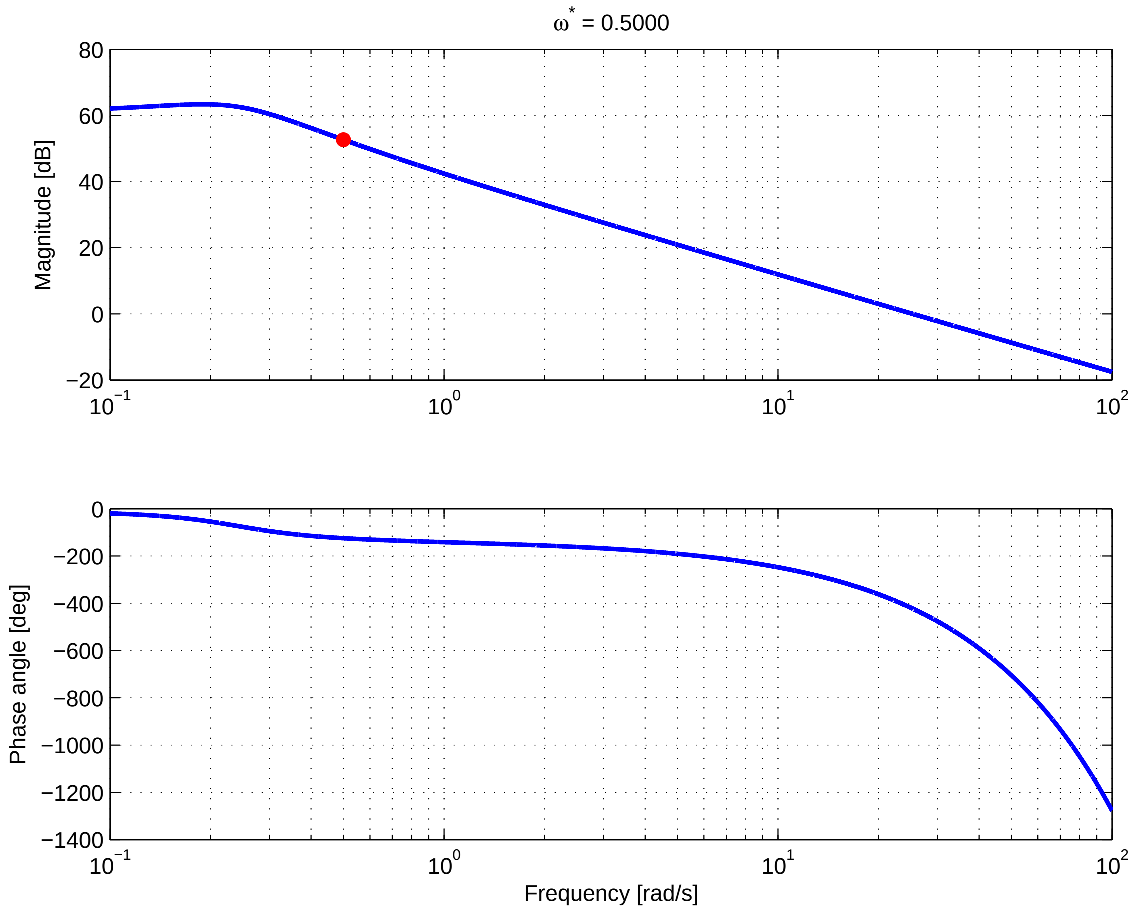
$$K = 10, \quad L = 0.2, \quad T = 8.5, \quad \alpha = 1.5 \tag{17}$$

and a FOPID controller with parameters

$$K_p = 100, \quad K_i = 2, \quad \lambda = 0.75, \quad K_d = 3, \quad \mu = 0.35. \tag{18}$$

The task is to locate the gain crossover frequency. This can be accomplished using the method described above.

Illustrative example: Newton iterations



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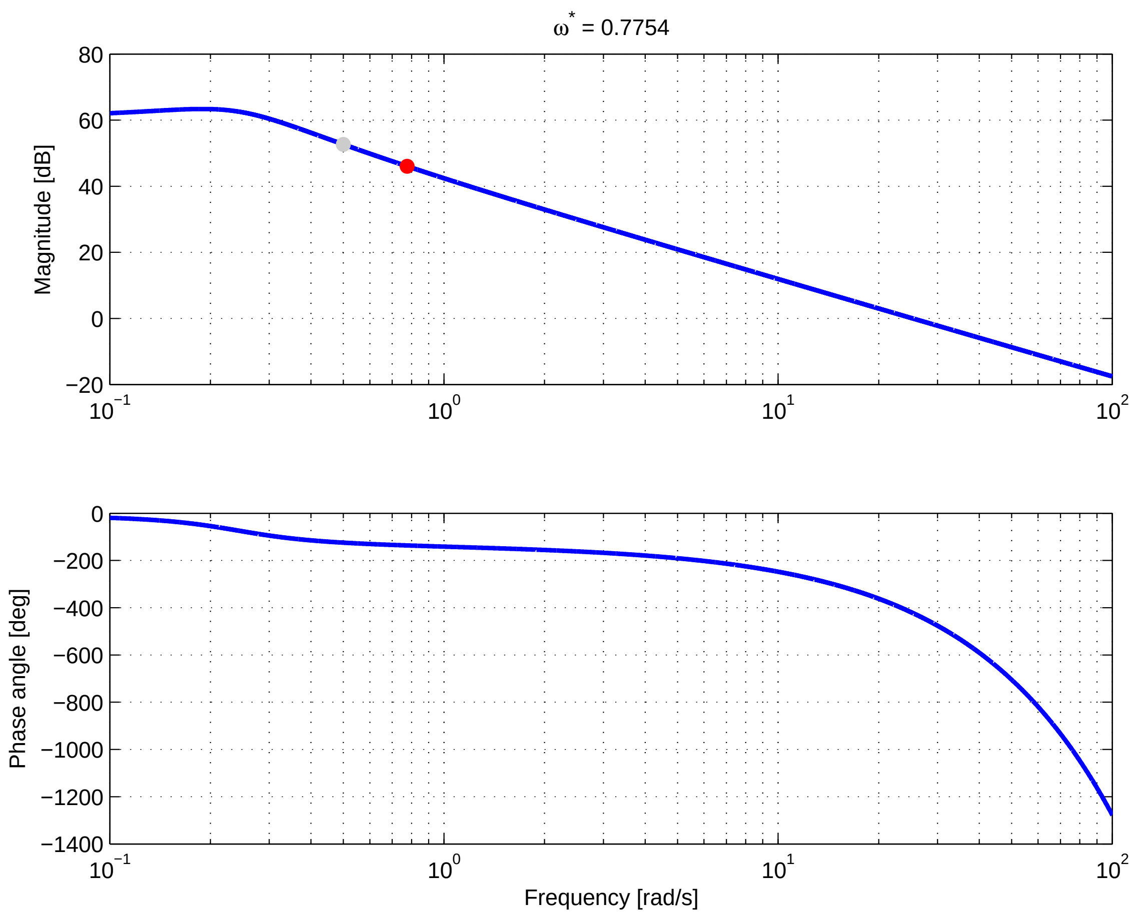

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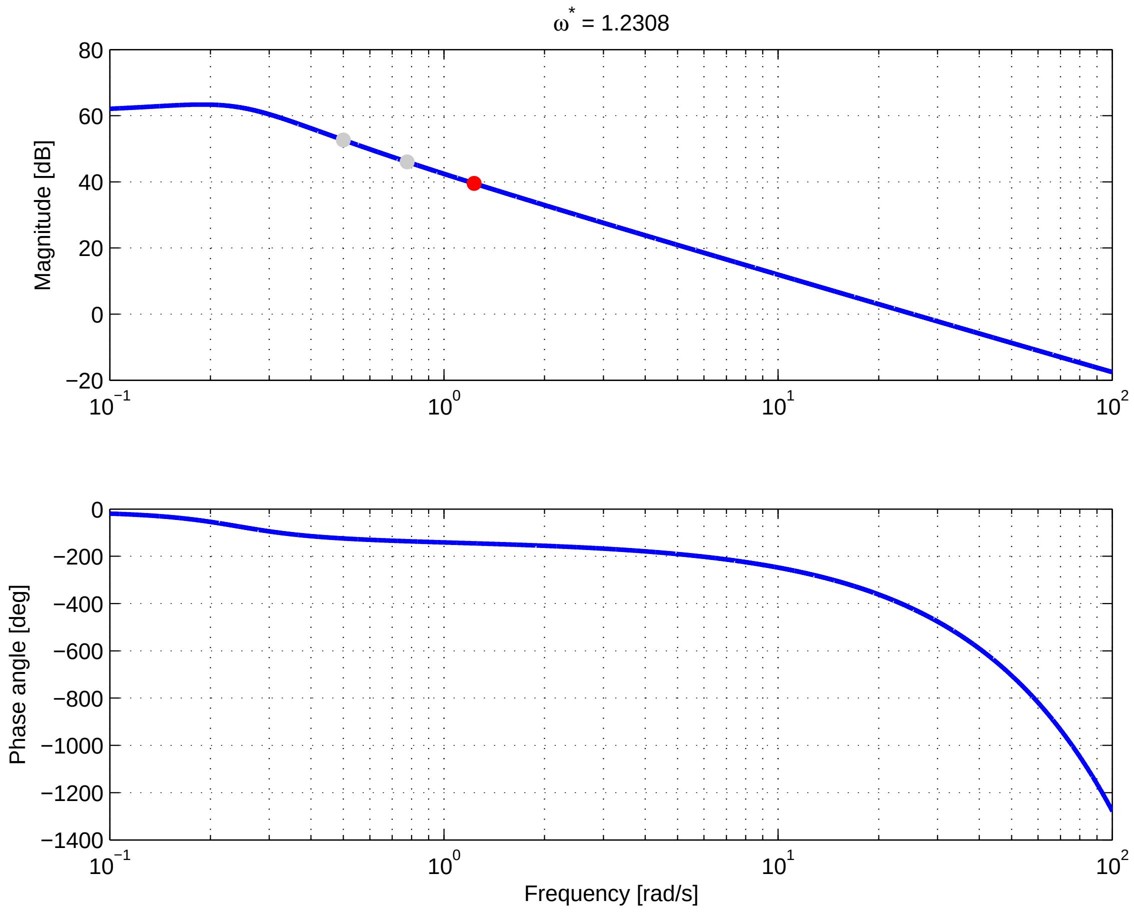

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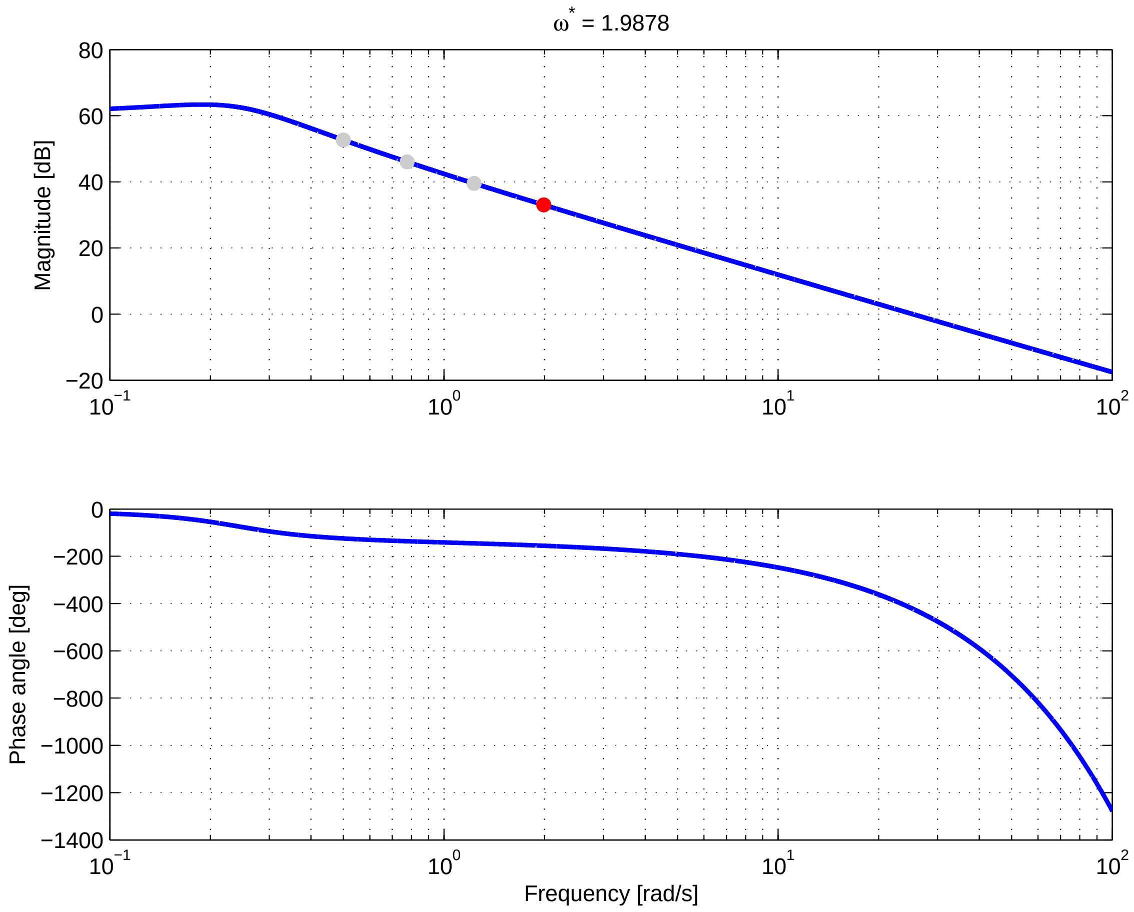

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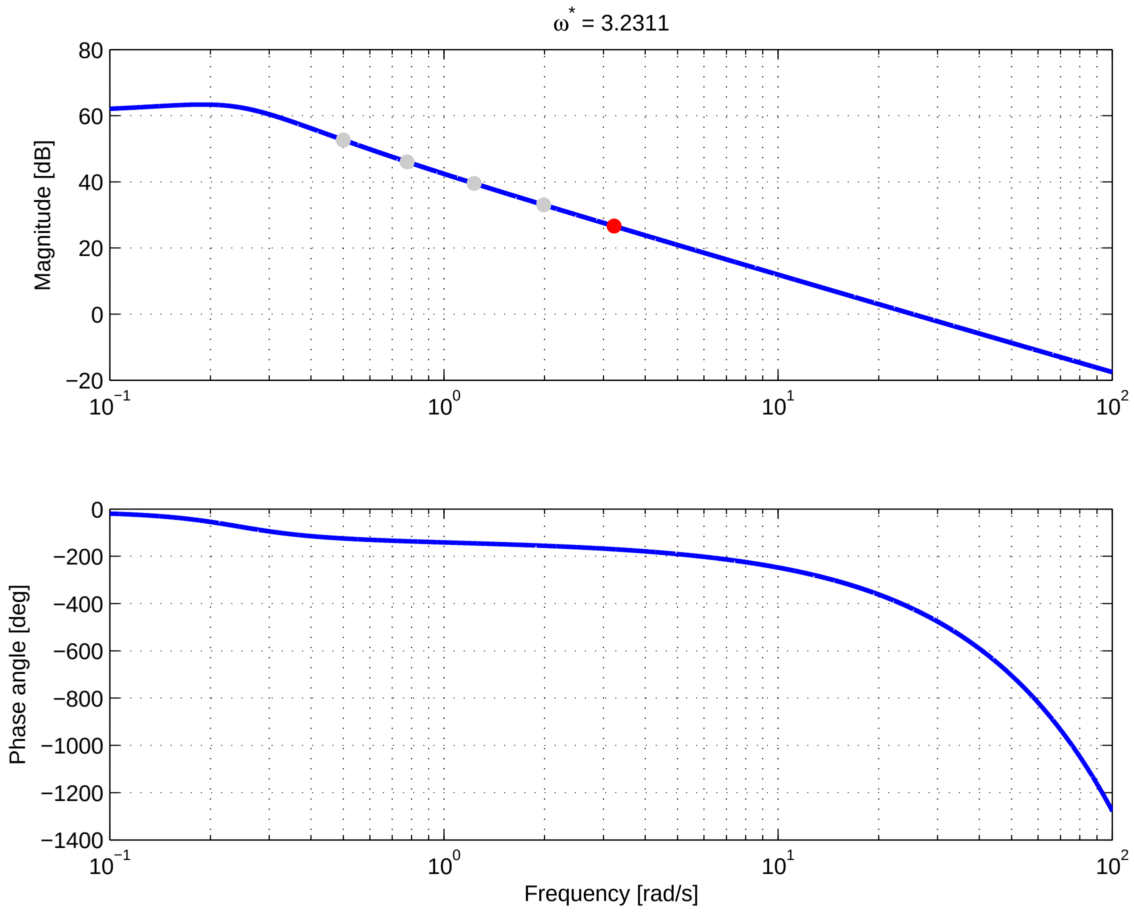

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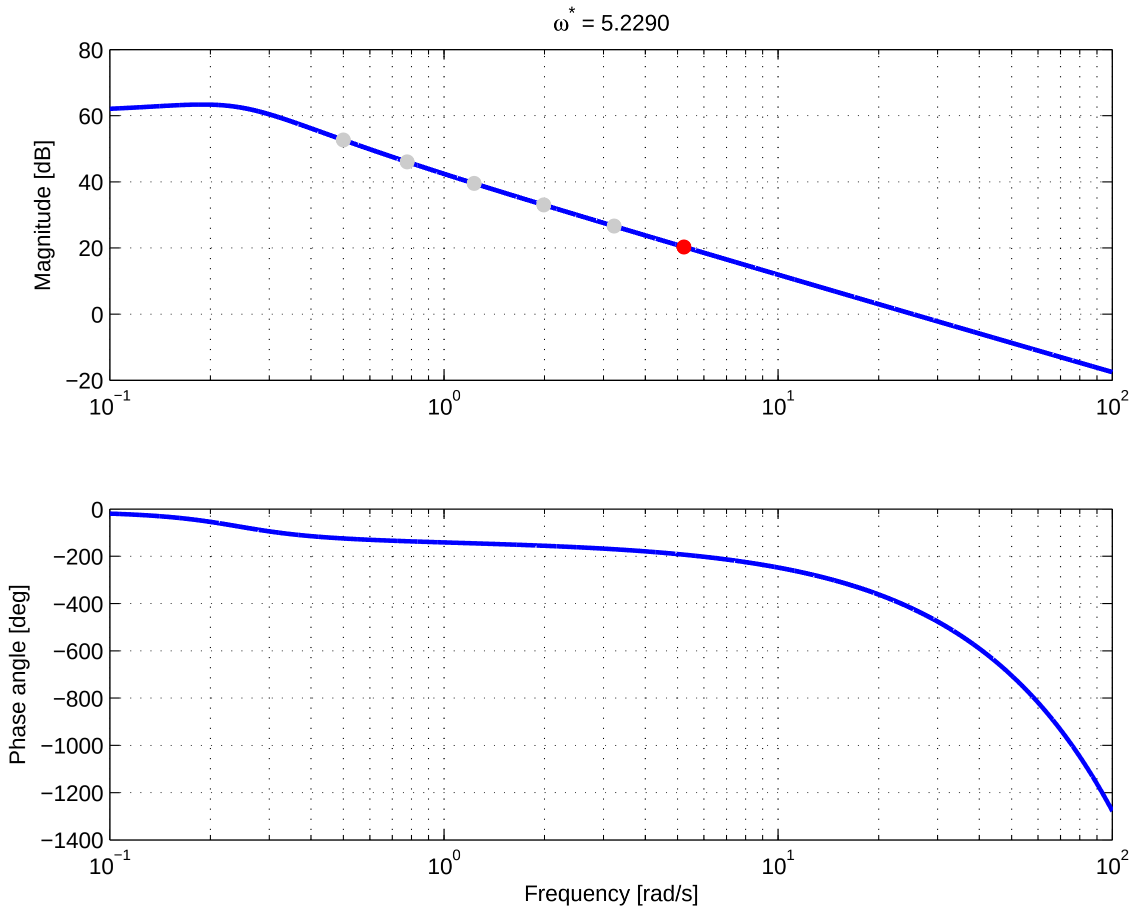

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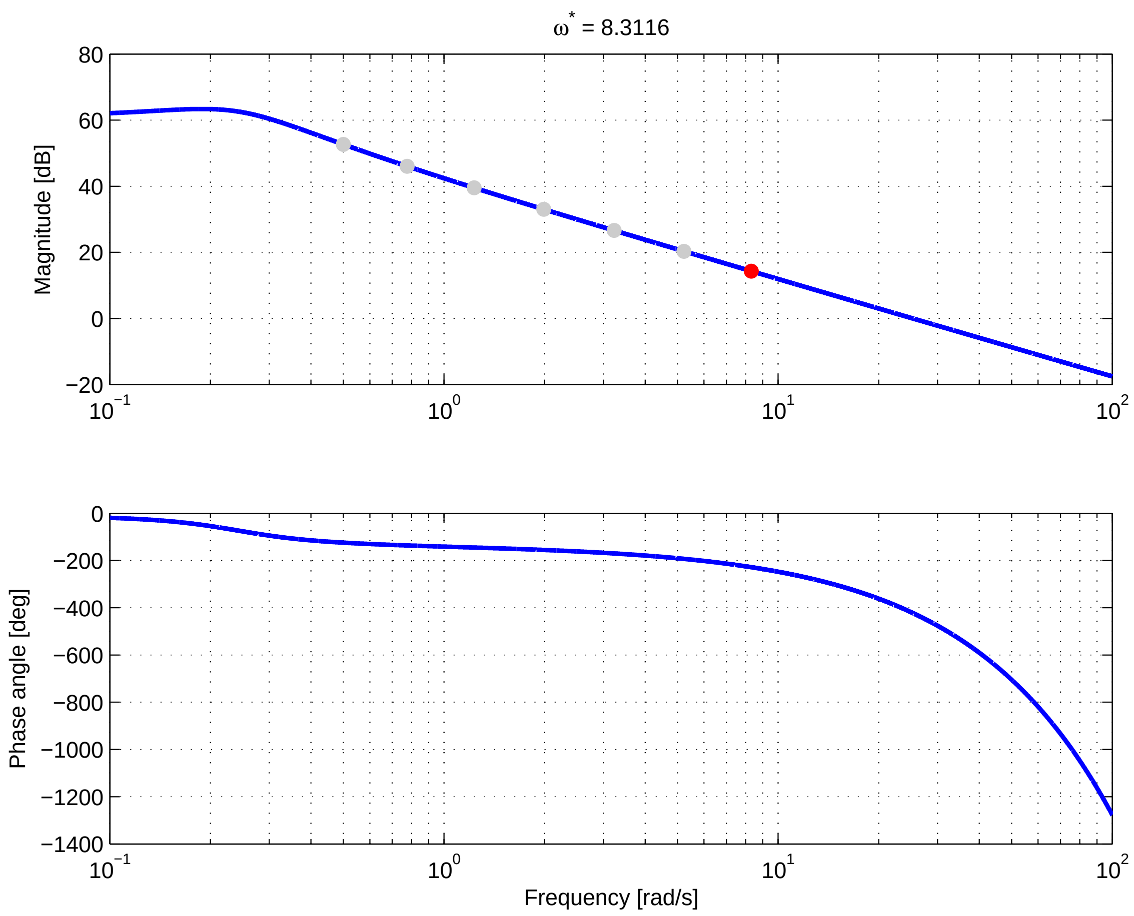

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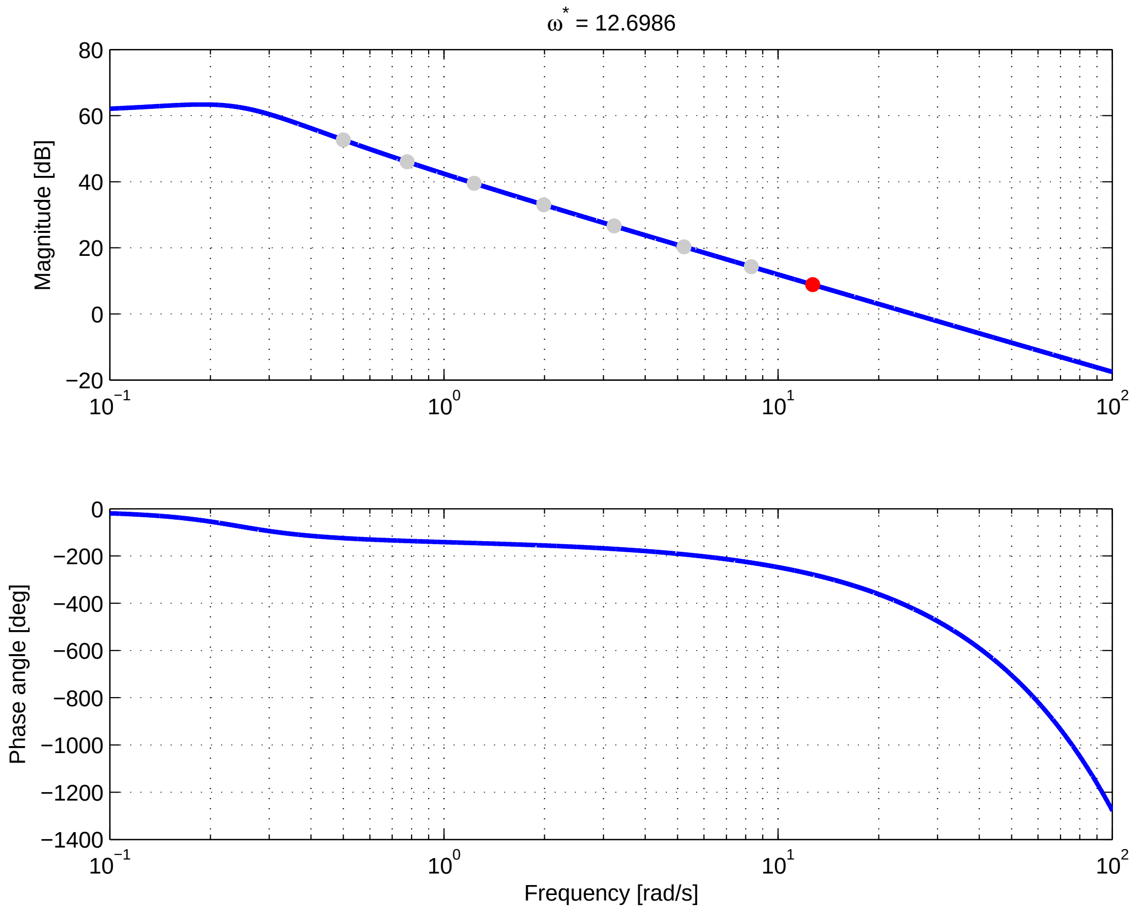

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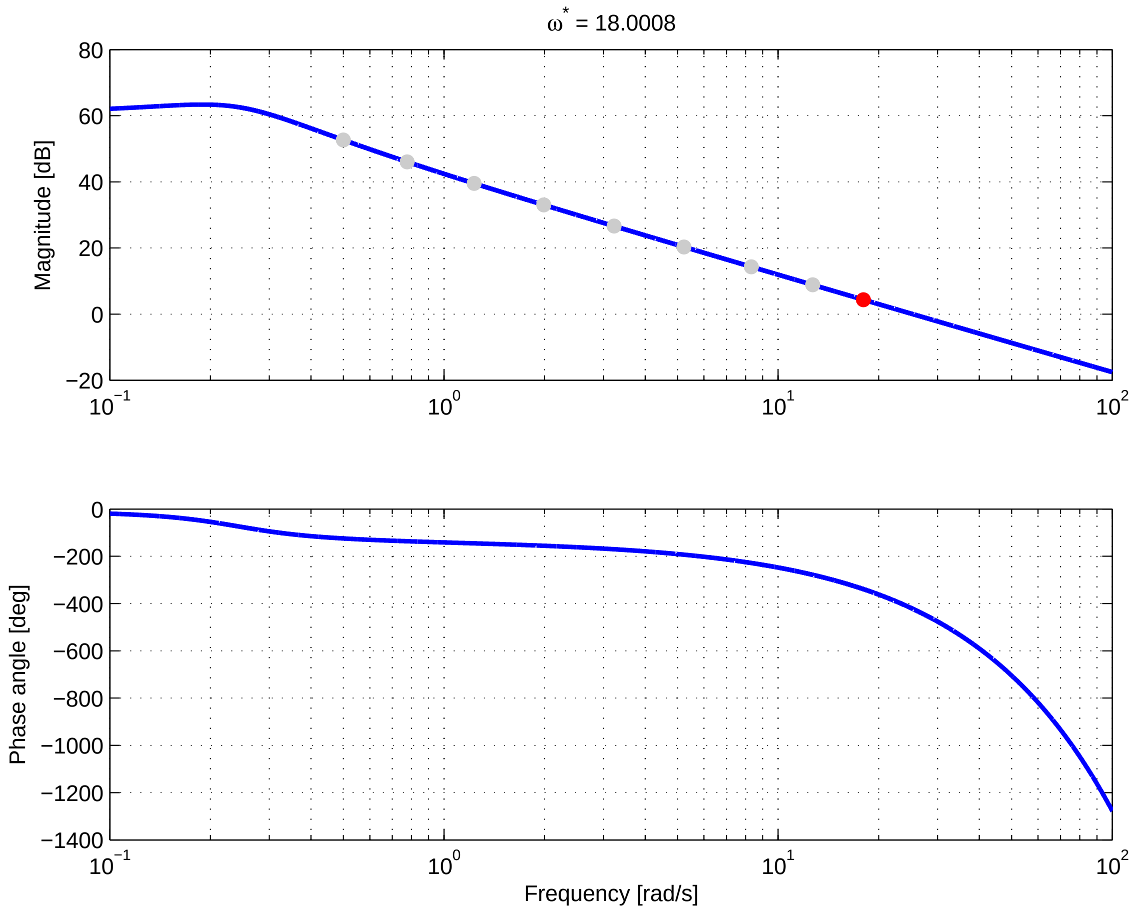

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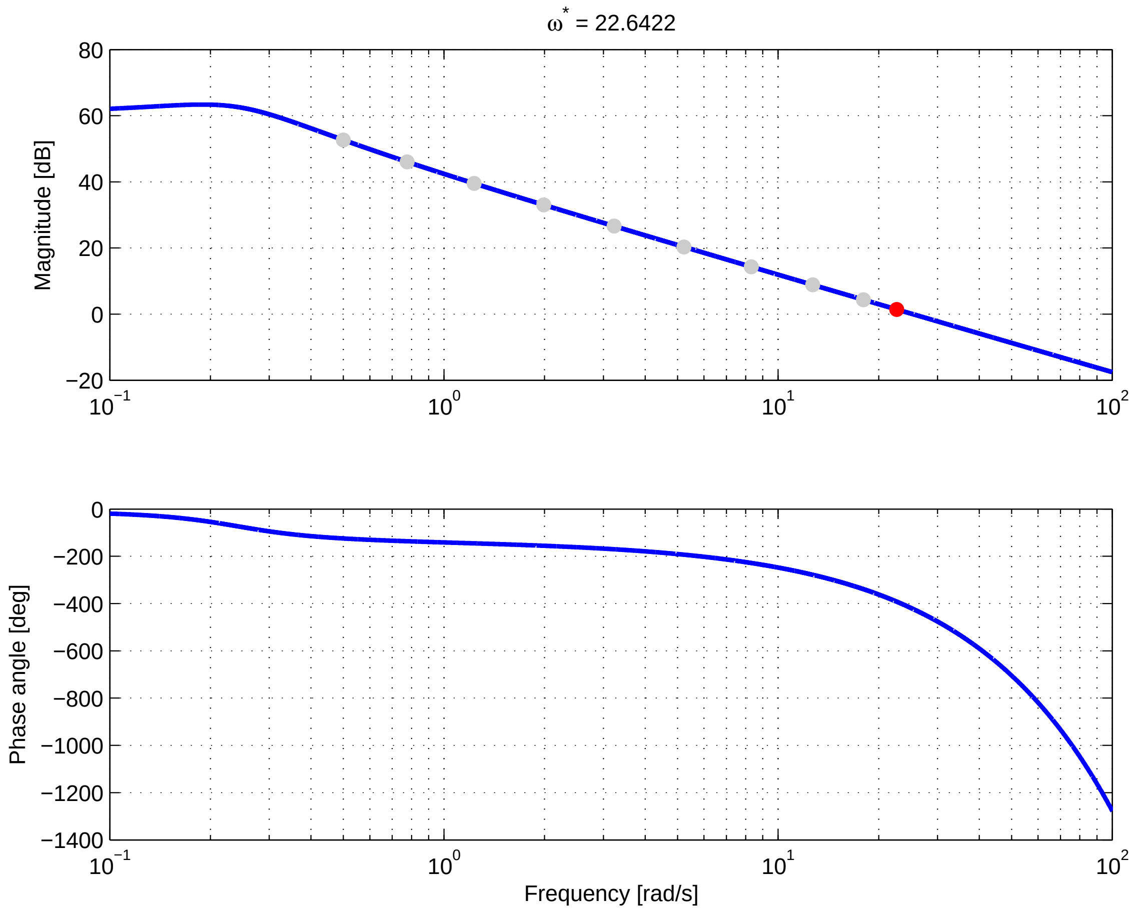

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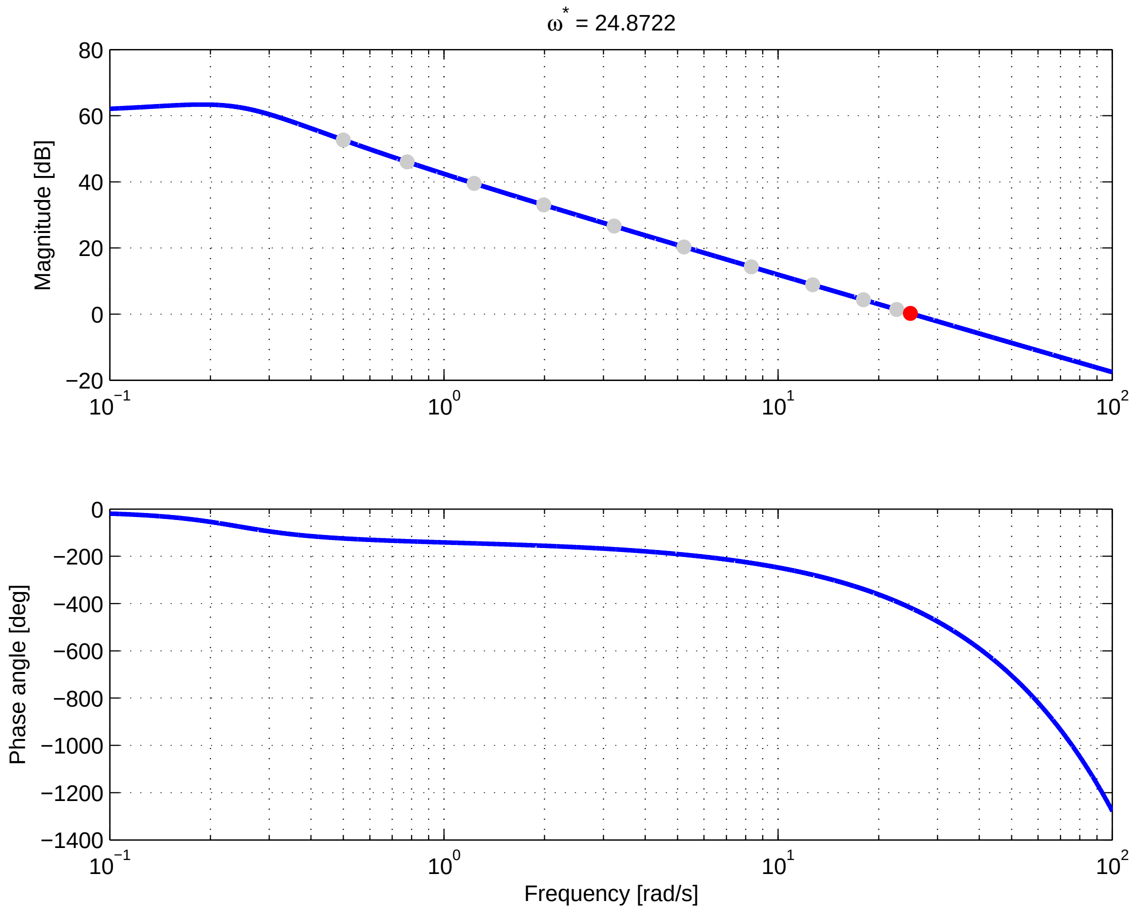

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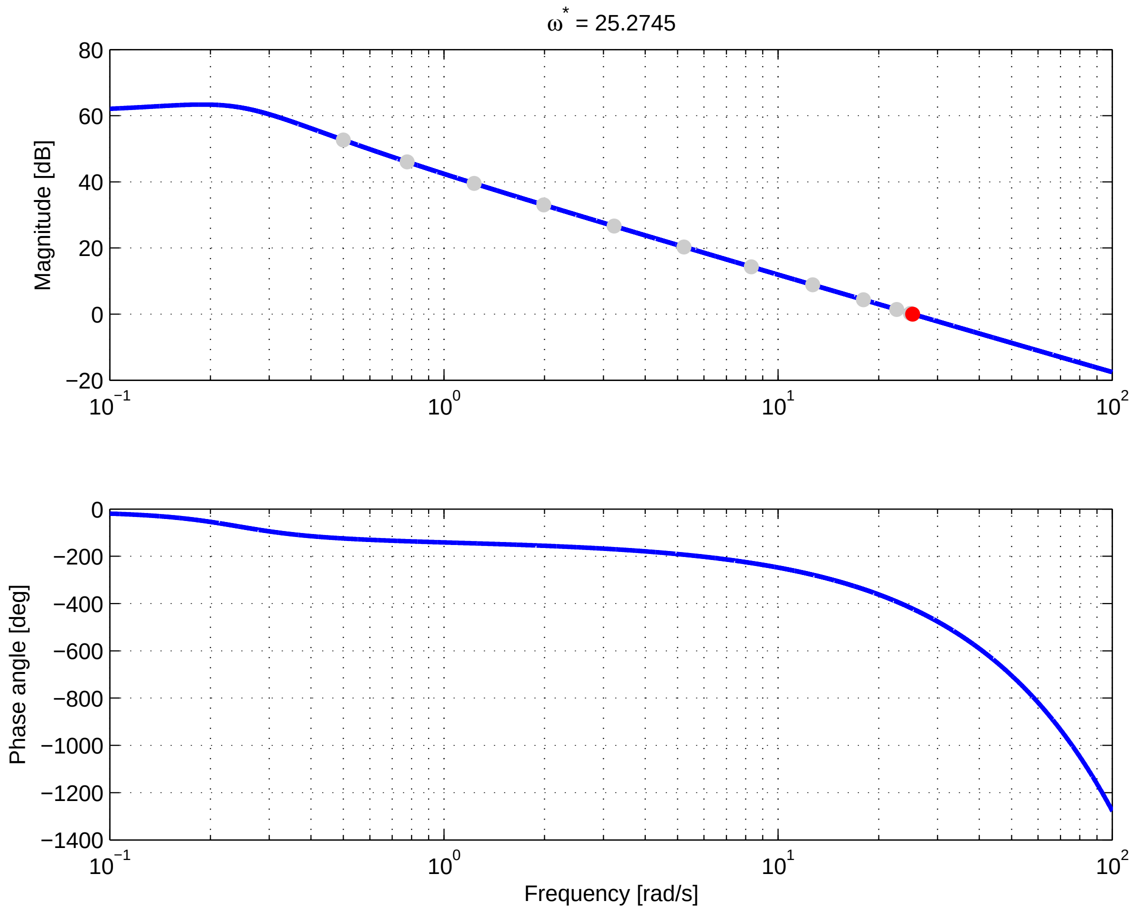

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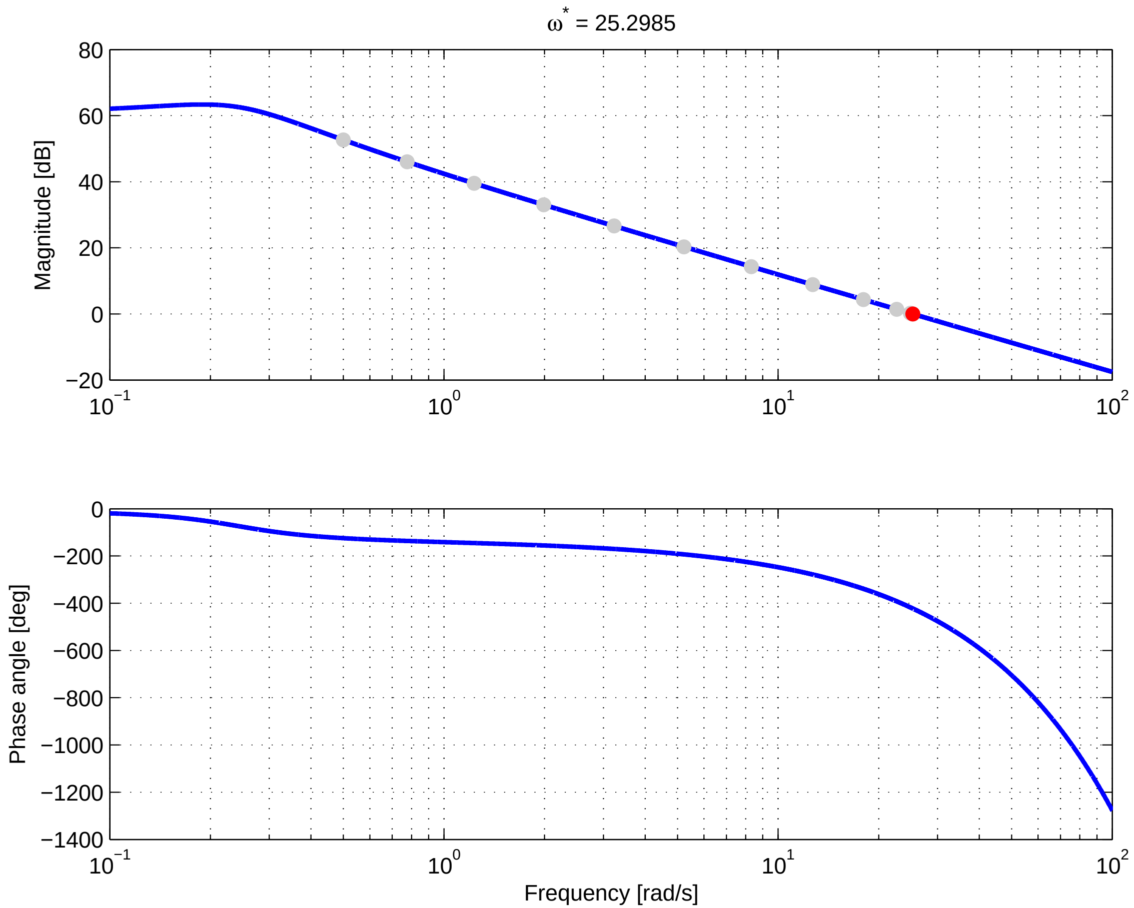

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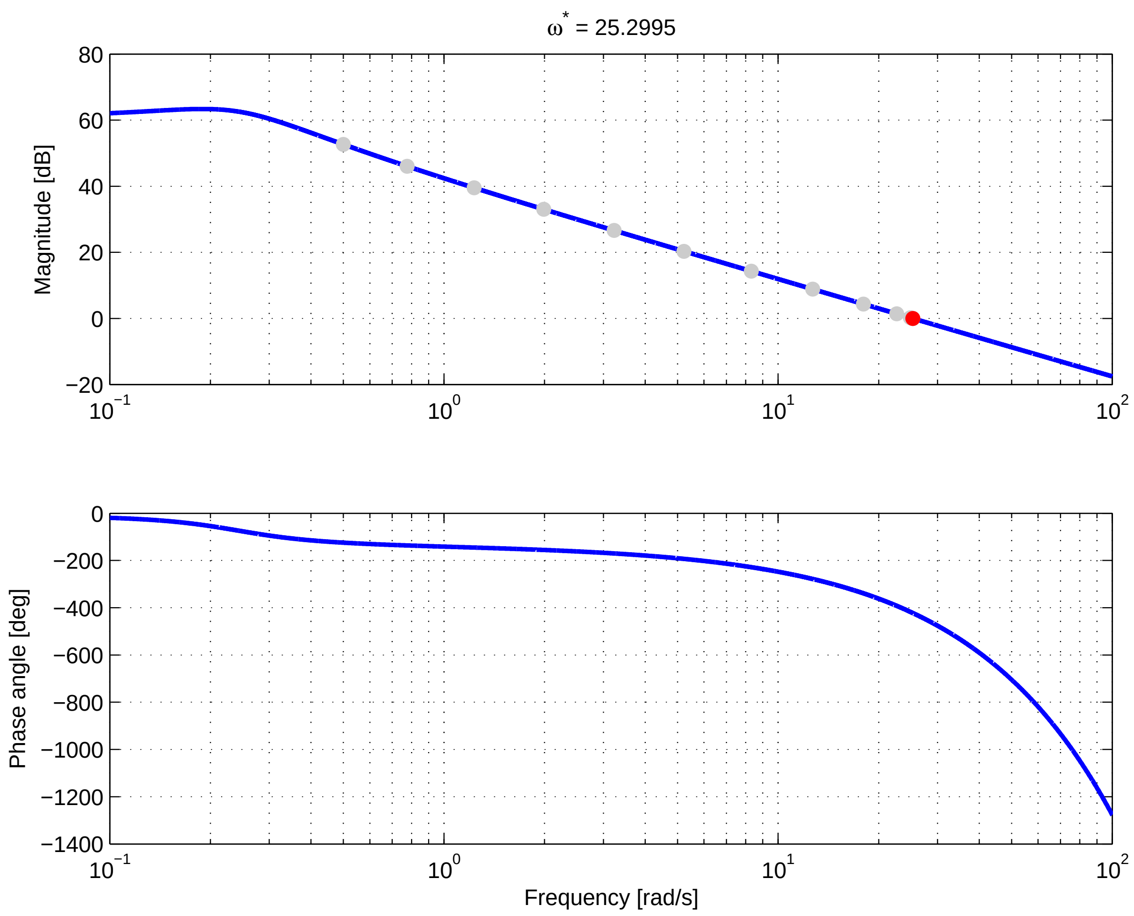

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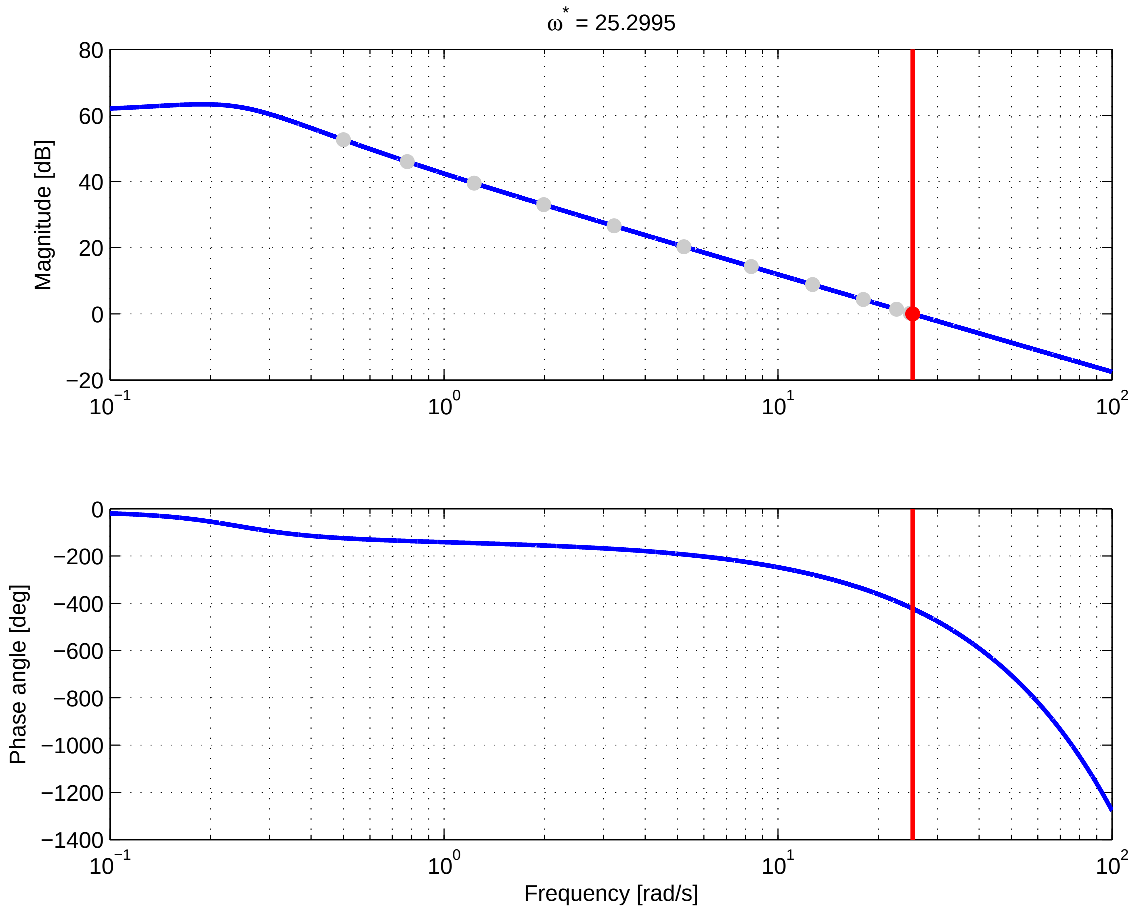


Proposed tuning methods: Fractional power sweep

Summary of the method:

1. Consider the parameters ω_u and K_c of the FFOPDT model in (7);
2. Compute the parameter set $\theta_G = \{K_p, K_i, K_d\}$ of a conventional PID-type controller;
3. Design a cost function $J(\cdot)$ based on desired system performance according to frequency-domain specifications;
4. Do a sweep of parameters in the set $\theta_P = \{\lambda, \mu\}$ within a predefined region, computing the specifications using the provided equations;
5. Choose the best controller according to $\min J(\cdot)$.

This method is justified because in case of a model (7) we have a fractional plant to control.

Proposed tuning methods: Full controller parameter optimization

In this work, we do not consider expensive time-domain simulations as basis for evaluating closed-loop system performance as opposed to our earlier work. This comes at the expense of neglecting nonlinear effects, such as actuator saturation. Only frequency-domain specifications are used. Consider, for example, the following weighted cost function

$$J = w_1 \left|\varphi_m - \tilde{\varphi}_m\right| + w_2 \left|G_m - \tilde{G}_m\right| + w_3 \left| \left(\mathrm{d} \arg \left(C(j\omega) G(j\omega) \right) / \mathrm{d}\omega \right)_{\omega=\omega_c} \right|. \quad (19)$$

The parameters of the controller $\theta = \{K_p, K_i, K_d, \lambda, \mu\}$ are then computed using a suitable optimization method.

Illustrative example: FOPI controller design

We consider here a FFOPDT model of a heating process. It is given by the following FFOPDT transfer function:

$$G(s) = \frac{66.16\mathrm{e}^{-1.93s}}{12.72s^{0.5}+1}. \tag{20}$$

We shall now design a fractional-order controller to control this plant using the method discussed above, first choosing three different sets of conventional PI tuning rules. Namely, the classical Ziegler-Nichols method, the Cohen-Coon method, and the AMIGO method.

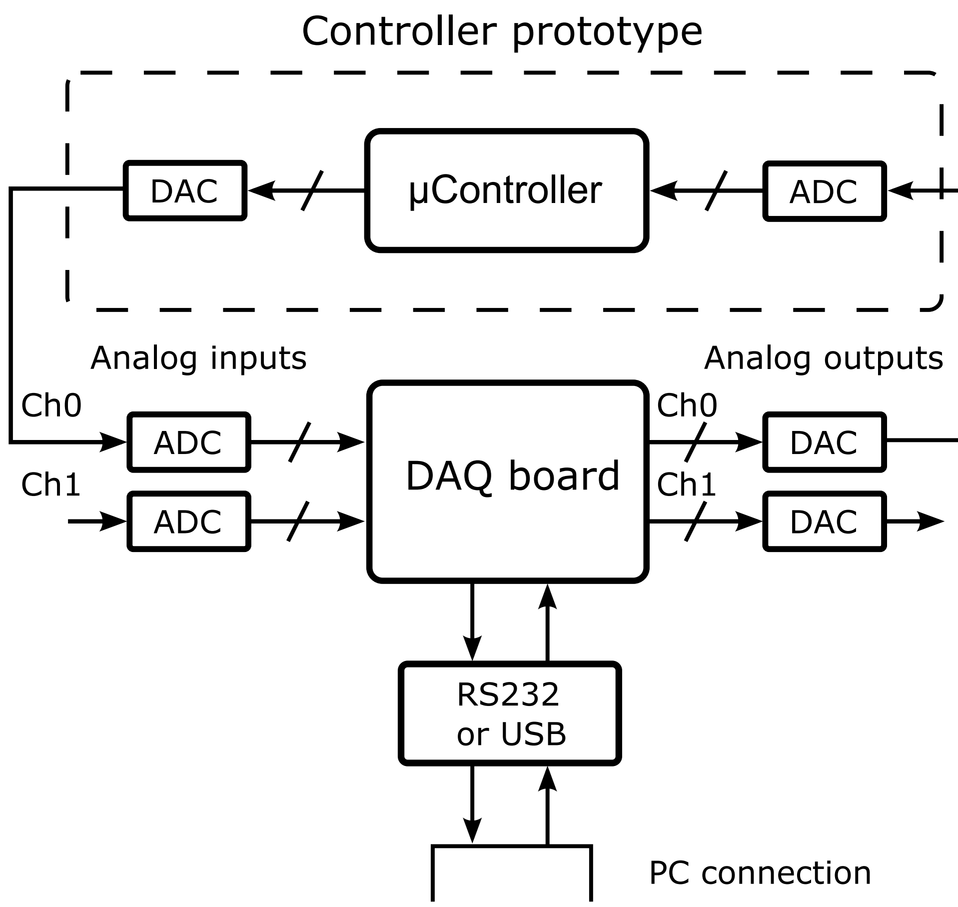
Since we are sweeping a single parameter, the weights of the cost in (19) are chosen such that $w_1 = 100/\pi$, $w_2 = 0$, $w_3 = 10$. The range of the integrator order sweep is selected as $\lambda \in [0.5, 1.5]$ with a step size of $\Delta\lambda = 0.05$. The desired phase margin is $\varphi_m = 75^\circ$. In addition, any λ that yields a control system with a gain margin such that $G_m < 2.5$ will be discarded.

Illustrative example: Tuning results

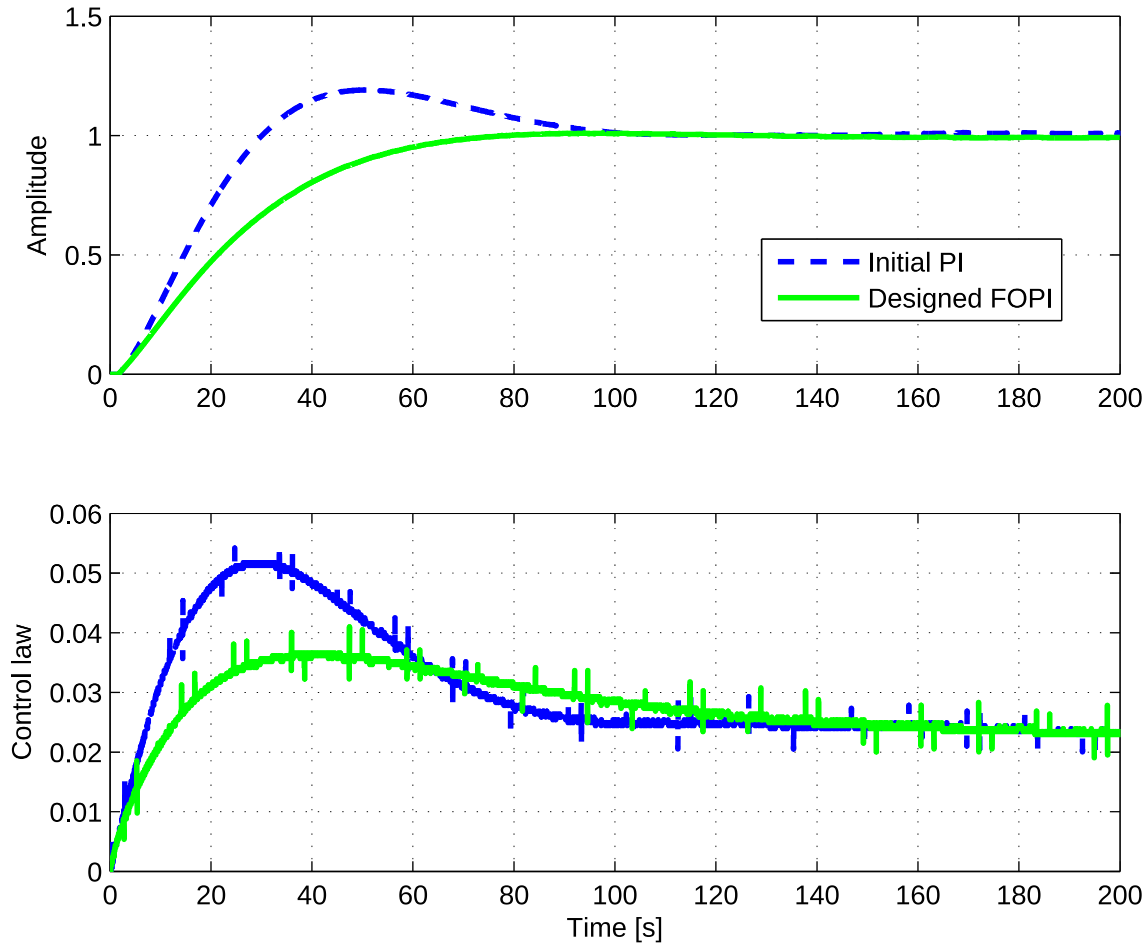
Method	K_p	K_i	λ^*	φ_m	G_m
Ziegler-Nichols	0.4059	2.2021	—	—	—
AMIGO	0.0024	0.0036	0.80	73.9°	30.7 dB
Cohen-Coon	0.0129	0.0375	0.55	74.7°	12.0 dB

As it can be seen, it was not possible to stabilize the control system in case of the Ziegler-Nichols tuning rules. However, in case of the AMIGO and Cohen-Coon tuning an suboptimal λ subject to given frequency-domain specifications, that is, the phase margin specification, was found.

Illustrative example: Hardware-in-the-Loop real-time experiment



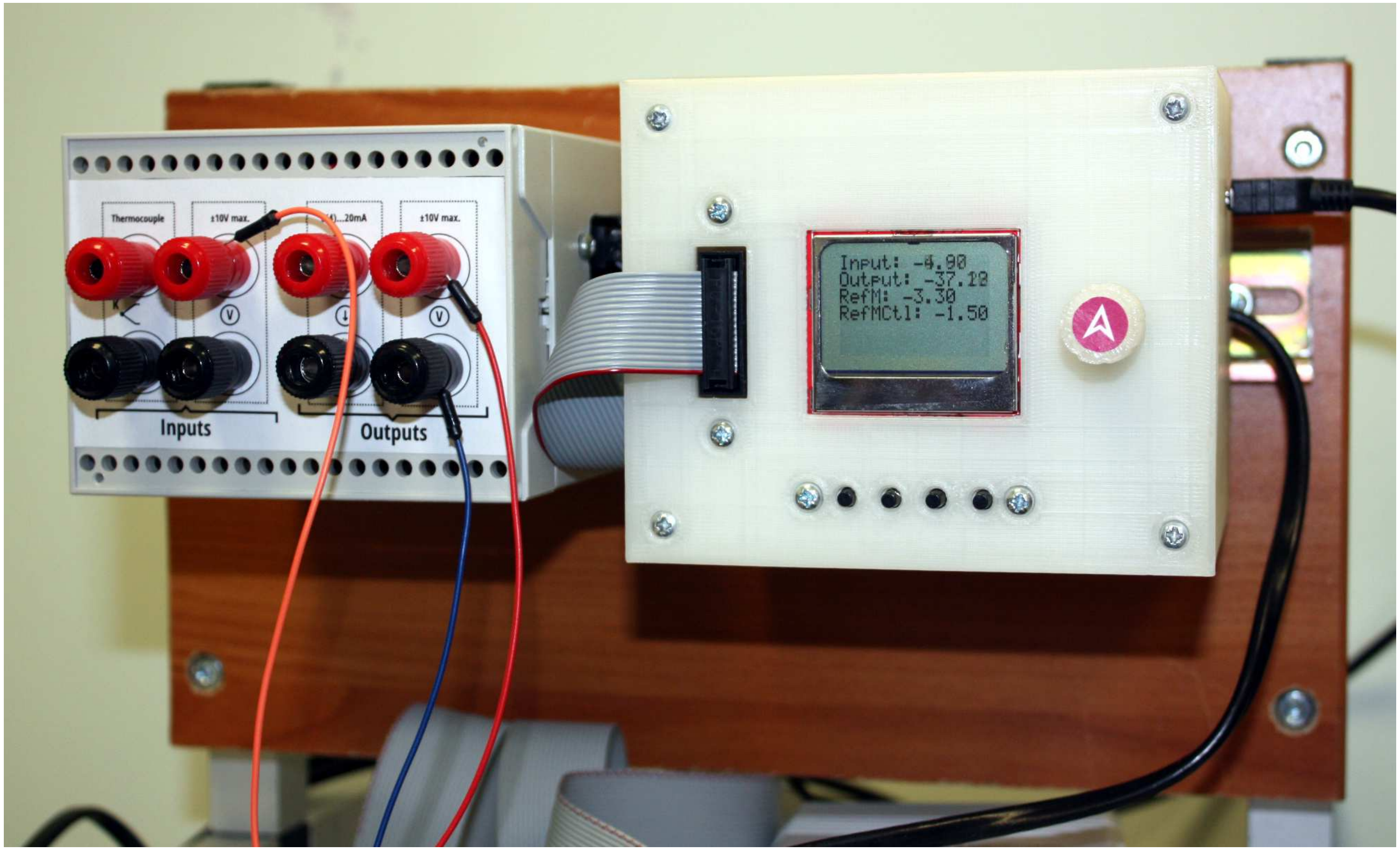
Illustrative example: AMIGO PI vs. designed FOPI



Conclusions and further perspectives

- In this work we have presented a set of relations that enable frequency-domain analysis of a control system comprising a FFOPDT plant and a FOPID controller.
- Using the frequency-domain performance criteria it is possible to design a suitable numerical optimization method to achieve prescribed robustness requirements of the control system.
- Since the involved computations are relatively simple, they may be implemented on embedded hardware platforms with limited computational capabilities. Thereby the real-time controller design issue is tackled.
- Future research should cover the optimization process with the goal of utilizing the discussed methods in a fractional-order PID controller with automatic tuning capabilities.

Future work: FOPID controller hardware prototype



FOMCON project: Fractional-order Modeling and Control

FOMCON

- Official website: `http://fomcon.net/`
- Toolbox for MATLAB available;
- An interdisciplinary project supported by the Estonian Doctoral School in ICT and Estonian Science Foundation grant nr. 8738.

Acknowledgement

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toetab Skype™

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`http://www.itakadeemia.ee/`

Questions?

Thank you for listening!

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