Lecture 2: The Direct-Quadrature-Zero (DQ0) Transformation

In the previous lecture we discussed the concept of time-varying phasor models (quasistatic models). We have seen that such models map sinusoidal signals to constants, and thus considerably simplify the analysis of AC power systems. Nevertheless, time-varying phasors are an approximation, and can only be used if variations in amplitudes and phases are relatively slow.

This lecture introduces the $dq0$ transformation, and shows how to use it to analyze linear networks. Similar to phasors, the $dq0$ transformation maps sinusoidal signals to constants, and therefore results in relatively simple dynamic models. However this mapping is accurate, and does not rely on any approximations. Therefore $dq0$ models may be viewed as a natural extension of time-varying phasor models, and are used extensively for modeling and analysis of fast transient phenomena in power systems.

| Table 1: The relative advantages of different dynamic models |
|-----------------|-----------------|-----------------|-----------------|
|                 | equilibrium    | small-          | fast            | nonsymmetric    |
|                 | points         | signal          | transients      | networks        |
| $abc$           | X              | X              | ✓              | ✓              |
| time-varying    | ✓              | ✓              | ✓              | X              |
| phasors         |                |                |                |                |
| $dq0$           | ✓              | ✓              | ✓              | see text       |

Basic definitions

The $dq0$ transformation and its inverse are defined as follows:

$$ T_{\theta} = \frac{2}{3} \begin{bmatrix} \cos(\theta) & \cos(\theta - \frac{2\pi}{3}) & \cos(\theta + \frac{2\pi}{3}) \\ -\sin(\theta) & -\sin(\theta - \frac{2\pi}{3}) & -\sin(\theta + \frac{2\pi}{3}) \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \tag{1} $$

$$ T_{\theta}^{-1} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 1 \\ \cos(\theta - \frac{2\pi}{3}) & -\sin(\theta - \frac{2\pi}{3}) & 1 \\ \cos(\theta + \frac{2\pi}{3}) & -\sin(\theta + \frac{2\pi}{3}) & 1 \end{bmatrix}, \tag{2} $$

where the angle $\theta$ is the reference angle or the reference phase. Direct multiplication of these matrices reveals that $T_{\theta}T_{\theta}^{-1} = T_{\theta}^{-1}T_{\theta} = I_{3\times3}$. Note that several variations of (1) are available in the literature.
The dq0 transformation maps three-phase signals in the abc reference frame to new quantities in a rotating dq0 reference frame. Denote \( x_{abc} = [x_a, x_b, x_c]^T \) and \( x_{dq0} = [x_d, x_q, x_0]^T \), then
\[
\begin{align*}
x_{dq0} &= T_{\theta} x_{abc}, \\
x_{abc} &= T_{\theta}^{-1} x_{dq0},
\end{align*}
\] (3)
where the subscripts \( d, q, \) and 0 represent the direct, quadrature and zero components.

The reference angle \( \theta \) is typically selected as follows:

✓ If there is an infinite bus in the system a typical choice is \( \theta = \omega_s t \), where \( \omega_s \) is the frequency of the infinite bus.

✓ If there is no infinite bus in the system, \( \theta \) is usually fixed to the rotor angle of one of the synchronous machines.

Different selections of the reference angle will be discussed in the following lectures.

A fundamental property of the dq0 transformation is that it maps balanced three-phase signals to constants. For instance, consider a three-phase voltage source modeled as
\[
\begin{align*}
v_a &= A \cos (\omega_s t), \\
v_b &= A \cos \left( \omega_s t - \frac{2\pi}{3} \right), \\
v_c &= A \cos \left( \omega_s t + \frac{2\pi}{3} \right).
\end{align*}
\] (4)
Applying the inverse transformation \( T_{\theta}^{-1} \) with \( \theta = \omega_s t \) leads to
\[
\begin{bmatrix}
v_a \\
v_b \\
v_c
\end{bmatrix} =
\begin{bmatrix}
\cos (\omega_s t) & -\sin (\omega_s t) & 1 \\
\cos (\omega_s t - \frac{2\pi}{3}) & -\sin (\omega_s t - \frac{2\pi}{3}) & 1 \\
\cos (\omega_s t + \frac{2\pi}{3}) & -\sin (\omega_s t + \frac{2\pi}{3}) & 1
\end{bmatrix}
\begin{bmatrix}
A \\
0 \\
0
\end{bmatrix},
\] (5)
and therefore \( v_d = A, v_q = 0, v_0 = 0 \). The sinusoidal signals in the abc reference frame are mapped to constant signals in the dq0 reference frame (see Fig. 1).

![Figure 1: Mapping of sinusoidal abc signals to constant dq0 signals.](image)
Modeling resistors, inductors, and capacitors

This section presents basic \(dq0\) models of linear passive components. We will use the following definitions:

- **Balanced** three-phase signals are sinusoidal signals with equal magnitudes, phase shifts of \(\pm 120^\circ\), and a sum of zero.

- A power network is **balanced** or **symmetrically configured** if balanced three-phase voltages at its ports result in balanced three-phase currents, and vice-versa. Two examples are shown in Fig. 2.

\[L_1 \quad \quad \quad L_2 \quad \quad \quad 2L_1\]
(a) Symmetrically configured network

\[L_3 \quad \quad \quad L_4 \quad \quad \quad L_5\]
(b) Network with a non-symmetric configuration

**Figure 2**: Example of symmetric and nonsymmetric configurations.

In this text we will use the term **balanced** when referring to signals, and **symmetrically configured** when referring to three-phase networks or circuits.

Assume a symmetrically configured three-phase resistor \(R\), which is modeled as

\[
\begin{bmatrix}
    v_a \\
    v_b \\
    v_c
\end{bmatrix} = R \begin{bmatrix}
    i_a \\
    i_b \\
    i_c
\end{bmatrix}.
\]

Multiplying both sides of the equation by the \(dq0\) transformation \(T_\theta\) (from the left) yields

\[
\begin{bmatrix}
    v_d \\
    v_q \\
    v_0
\end{bmatrix} = R \begin{bmatrix}
    i_d \\
    i_q \\
    i_0
\end{bmatrix}.
\]

This is the \(dq0\) model of a symmetrically configured three-phase resistor.

Now assume a symmetrically configured three-phase inductor \(L\), which is modeled as

\[
\begin{bmatrix}
    v_a \\
    v_b \\
    v_c
\end{bmatrix} = L \frac{d}{dt} \begin{bmatrix}
    i_a \\
    i_b \\
    i_c
\end{bmatrix}.
\]

The identity \([i_a, i_b, i_c]^T = T_\theta^{-1}[i_d, i_q, i_0]^T\) leads to

\[
\begin{bmatrix}
    v_a \\
    v_b \\
    v_c
\end{bmatrix} = L \frac{d}{dt} \left( T_\theta^{-1} \begin{bmatrix}
    i_d \\
    i_q \\
    i_0
\end{bmatrix} \right),
\]

and the derivative product rule yields

\[
\begin{bmatrix}
    v_a \\
    v_b \\
    v_c
\end{bmatrix} = L \left( \frac{d}{dt} T_\theta^{-1} \right) \begin{bmatrix}
    i_d \\
    i_q \\
    i_0
\end{bmatrix} + LT_\theta^{-1} \frac{d}{dt} \begin{bmatrix}
    i_d \\
    i_q \\
    i_0
\end{bmatrix}.
\]
By direct computation it can be verified that
\[
\frac{d}{dt} T^{-1}_\theta = -T^{-1}_\theta W
\] (11)
with
\[
W = \begin{bmatrix} 0 & \frac{d}{dt} \theta & 0 \\ \frac{d}{dt} \theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\] (12)
Substituting these expressions in (10) yields
\[
\begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = -LT^{-1}_\theta W \begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} + LT^{-1}_\theta \frac{d}{dt} \begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix},
\] (13)
and by multiplying both sides of the equation from the left by \(T_\theta\) we have
\[
\begin{bmatrix} v_d \\ v_q \\ v_0 \end{bmatrix} = -LW \begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} + \frac{1}{L} \begin{bmatrix} v_d \\ v_q \\ v_0 \end{bmatrix},
\] (14)
which is equivalent to
\[
\frac{d}{dt} \begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} = W \begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} + \frac{1}{L} \begin{bmatrix} v_d \\ v_q \\ v_0 \end{bmatrix}.
\] (15)
This is the \(dq_0\) model of a symmetrically configured inductor. The explicit expressions are
\[
\begin{align*}
\frac{d}{dt} i_d &= \frac{d}{dt} \theta i_q + \frac{1}{L} v_d, \\
\frac{d}{dt} i_q &= -\frac{d}{dt} \theta i_d + \frac{1}{L} v_q, \\
\frac{d}{dt} i_0 &= \frac{1}{L} v_0.
\end{align*}
\] (16)
Note that \(i_d\) affects the dynamics of \(i_q\) and vice-versa. Similarly, the dynamic model of a symmetrically configured capacitor \(C\) is
\[
\frac{d}{dt} \begin{bmatrix} v_d \\ v_q \\ v_0 \end{bmatrix} = W \begin{bmatrix} v_d \\ v_q \\ v_0 \end{bmatrix} + \frac{1}{C} \begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix}.
\] (17)

**Power and energy in terms of \(dq_0\) quantities**

Consider a general three-phase unit as described in Fig. 3.
The instantaneous power flowing from the unit into the network at time \(t\) is
\[
p_{3\phi}(t) = v_a(t)i_a(t) + v_b(t)i_b(t) + v_c(t)i_c(t).
\] (18)
Rewrite this equation as
\[
p_{3\phi} = \begin{bmatrix} v_a & v_b & v_c \end{bmatrix} \begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} = \left( T^{-1}_\theta \begin{bmatrix} v_d \\ v_q \\ v_0 \end{bmatrix} \right)^T T^{-1}_\theta \begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix}
\] (19)
\[= \begin{bmatrix} v_d & v_q & v_0 \end{bmatrix} \left( T^{-1}_\theta \right)^T T^{-1}_\theta \begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} \]
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Figure 3: Three-phase unit connected to the network.

and use

$$(T_\theta^{-1})^T T_\theta^{-1} = \frac{3}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$ \hspace{1cm} (20)$$

to obtain

$$p_{3\phi} = \frac{3}{2} \begin{bmatrix} v_d & v_q & v_0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} = \frac{3}{2} (v_d i_d + v_q i_q + 2v_0 i_0).$$ \hspace{1cm} (21)

This expression defines the three-phase instantaneous power in terms of $dq0$ quantities.

Similarly, assume a symmetrically configured three-phase inductor, with currents $i_a, i_b, i_c$ as shown in Fig. 4.

Figure 4: Symmetrically configured three-phase inductor.

The total energy stored in the inductor is

$$E = \frac{1}{2} L \left( i_a^2 + i_b^2 + i_c^2 \right),$$ \hspace{1cm} (22)

which may be written as

$$E = \frac{1}{2} L \begin{bmatrix} i_a & i_b & i_c \end{bmatrix}^T = \frac{1}{2} L \begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} \begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} i_d & i_q & i_0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix}.$$ \hspace{1cm} (23)

Again use the identity

$$(T_\theta^{-1})^T T_\theta^{-1} = \frac{3}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$ \hspace{1cm} (24)

to obtain

$$E = \frac{3}{4} L \begin{bmatrix} i_d & i_q & i_0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix},$$ \hspace{1cm} (25)
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which leads to

$$E = \frac{3}{4} L \left( i_d^2 + i_q^2 + 2 i_0^2 \right).$$

(26)

Similarly, the energy stored in a symmetrically configured three-phase capacitor $C$ is

$$E = \frac{3}{4} C \left( v_d^2 + v_q^2 + 2 v_0^2 \right).$$

(27)

The following analysis shows that the energy derivative with respect to time is power, as expected. Consider the circuit in Fig. 4. Application of the chain rule yields

$$\frac{d}{dt} E = \frac{3}{2} L \left( i_d \frac{di_d}{dt} + i_q \frac{di_q}{dt} + 2 i_0 \frac{di_0}{dt} \right),$$

(28)

and based on the dynamic model of the inductor in (16) we have

$$\frac{d}{dt} E = \frac{3}{2} \left[ i_d \left( \frac{d\theta}{dt} i_d + \frac{1}{L} (v_{d,1} - v_{d,2}) \right) + i_q \left( -\frac{d\theta}{dt} i_d + \frac{1}{L} (v_{q,1} - v_{q,2}) \right) + 2 i_0 \left( \frac{1}{L} (v_{0,1} - v_{0,2}) \right) \right].$$

(29)

This expression can be simplified as

$$\frac{d}{dt} E = \frac{3}{2} \left( v_{d,1} i_d + v_{q,1} i_q + 2 v_{0,1} i_0 \right) - \frac{3}{2} \left( v_{d,2} i_d + v_{q,2} i_q + 2 v_{0,2} i_0 \right),$$

(30)

which is identical to

$$\frac{d}{dt} E = p_1 - p_2.$$

(31)

The change in stored energy is equal to the sum of powers flowing into the inductor.

**Modeling linear circuits**

We will now extend the discussion in the previous section and show how to construct $dq0$ models of general three-phase circuits.

Consider a three-phase circuit composed of inductors, capacitor, resistors, voltage sources and current sources. Let $x$ represent the state vector of this circuit in the $abc$ reference frame, and use the compact notation $x_{abc} = [x_{a,1}, x_{b,1}, x_{c,1}, \ldots, x_{a,m}, x_{b,m}, x_{c,m}]^T$. The circuit dynamics may be expressed as

$$\frac{d}{dt} x_{abc} = A x_{abc} + B u.$$

(32)

Define the composite $dq0$ transformation and its inverse as

$$\Lambda_\theta = \begin{bmatrix} T_\theta & 0 \\ \vdots & \ddots \\ 0 & T_\theta \end{bmatrix}, \quad \Lambda_\theta^{-1} = \begin{bmatrix} T_\theta^{-1} & 0 \\ \vdots & \ddots \\ 0 & T_\theta^{-1} \end{bmatrix},$$

(33)

such that $x_{dq0} = \Lambda_\theta x_{abc}$ and $x_{abc} = \Lambda_\theta^{-1} x_{dq0}$. Substitute these definitions into (32) to get

$$\frac{d}{dt} \left( \Lambda_\theta^{-1} x_{dq0} \right) = A \Lambda_\theta^{-1} x_{dq0} + B u,$$

(34)
\[
\frac{d}{dt} \left( \Lambda_{\theta}^{-1} \right) x_{dq0} + \Lambda_{\theta}^{-1} \frac{d}{dt} x_{dq0} = A \Lambda_{\theta}^{-1} x_{dq0} + Bu. \tag{35}
\]

It can be verified by direct calculations that
\[
\frac{d}{dt}(\Lambda_{\theta}^{-1}) = -\Lambda_{\theta}^{-1} W_c, \tag{36}
\]

where
\[
W_c = \begin{bmatrix}
W & 0 \\
\vdots & \ddots \\
0 & W
\end{bmatrix}, \quad W = \begin{bmatrix}
0 & \frac{d\theta}{dt} & 0 \\
-\frac{d\theta}{dt} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}. \tag{37}
\]

Substitute the expression for \(\frac{d}{dt}(\Lambda_{\theta}^{-1})\) into (35) to obtain
\[
-\Lambda_{\theta}^{-1} W_c x_{dq0} + \Lambda_{\theta}^{-1} \frac{d}{dt} x_{dq0} = A \Lambda_{\theta}^{-1} x_{dq0} + Bu, \tag{38}
\]

and multiply from the left by \(\Lambda_{\theta}\) to get
\[
\frac{d}{dt} x_{dq0} = \left( \Lambda_{\theta} A \Lambda_{\theta}^{-1} + W_c \right) x_{dq0} + \Lambda_{\theta} Bu. \tag{39}
\]

This equation describes the circuit dynamics based on \(dq0\) quantities. In general the expression on the right depends on \(\theta(t)\), and therefore this model does not have well-defined equilibrium points. However, for the special case of symmetrically configured networks, it is typically true that \(\Lambda_{\theta} A = A \Lambda_{\theta}\), so equation (38) takes the form
\[
\frac{d}{dt} x_{dq0} = (A + W_c) x_{dq0} + \Lambda_{\theta} Bu. \tag{40}
\]

Under the typical assumption that the input \(\Lambda_{\theta} Bu\) is constant, the right hand side of this equation does not include \(\theta(t)\), and therefore this model has well-defined equilibrium points, and can be analyzed using standard tools.

As an example, consider the circuit in Fig. 5.

![Figure 5: Example—symmetrically configured RL transmission line.](image)

The dynamic model in the \(abc\) reference frame is
\[
\frac{d}{dt} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} = -\frac{R}{L} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} + \frac{1}{L} \begin{bmatrix} v_{a,1} - v_{a,2} \\ v_{b,1} - v_{b,2} \\ v_{c,1} - v_{c,2} \end{bmatrix}. \tag{41}
\]

In this example \(\Lambda_{\theta} = T_\theta\), and therefore \(W_c = W\). In addition it may be verified that \(T_\theta A = AT_\theta\). Based on (39) the dynamic model in the \(dq0\) reference frame is
\[
\frac{d}{dt} i_{dq0} = (A + W) i_{dq0} + T_\theta Bu, \tag{42}
\]

which yields
\[
\frac{d}{dt} \begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} = (A + W) \begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} + T_\theta B \begin{bmatrix} v_{a,1} - v_{a,2} \\ v_{b,1} - v_{b,2} \\ v_{c,1} - v_{c,2} \end{bmatrix}. \tag{43}
\]
or equivalently
\[
\frac{d}{dt} \begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} = \left( -\frac{R}{L} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & \frac{d}{dt} \theta & 0 \\ \frac{d}{dt} \theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} + \frac{1}{L} T_\theta \begin{bmatrix} v_{a,1} - v_{a,2} \\ v_{b,1} - v_{b,2} \\ v_{c,1} - v_{c,2} \end{bmatrix}. \tag{43}
\]

This last equation may be written more compactly as
\[
\frac{d}{dt} \begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} = \left( -\frac{R}{L} \begin{bmatrix} \frac{d}{dt} \theta & 0 \\ 0 & -\frac{R}{L} \end{bmatrix} \right) \begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} + \frac{1}{L} \begin{bmatrix} v_{d,1} - v_{d,2} \\ v_{q,1} - v_{q,2} \\ v_{0,1} - v_{0,2} \end{bmatrix}. \tag{44}
\]

In addition, if the reference angle is \( \theta = \omega_s t \) then
\[
\frac{d}{dt} \begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} = \left( -\frac{R}{L} \begin{bmatrix} \omega_s & 0 \\ -\omega_s & -\frac{R}{L} \end{bmatrix} \right) \begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} + \frac{1}{L} \begin{bmatrix} v_{d,1} - v_{d,2} \\ v_{q,1} - v_{q,2} \\ v_{0,1} - v_{0,2} \end{bmatrix}. \tag{45}
\]

The resulting model is linear and time-invariant.

How does this model change if the circuit is not symmetrically configured?

Consider a modified circuit in which the inductances are \( L, L, \) and \( 2L \). In this case the equality \( \Lambda_\theta A = \Lambda_\theta \Lambda_{\theta} \) no longer holds, and therefore the dynamic model is nonlinear and time-varying, since the term \( \Lambda_\theta A\Lambda_{\theta}^{-1} \) varies with \( \theta(t) \). In this example the \( dq0 \) transformation does not offer any obvious advantage in comparison to direct analysis in the \( abc \) reference frame.

As another example consider the circuit in Fig. 6. The voltage sources on the left represent the secondary side of a transformer, and the voltage sources on the right represent the primary side of another transformer. The objective here is to study the effects of \( R_{g,1} \) and \( R_{g,2} \) on the system dynamics.

![Figure 6: Example — a linear three-phase circuit.](image)

The model in the \( abc \) reference frame is
\[
L \frac{d}{dt} i_a = -R_i a - (R_{g,1} + R_{g,2})(i_a + i_b + i_c) + (v_{a,1} - v_{a,2}),
\]
\[
L \frac{d}{dt} i_b = -R_i b - (R_{g,1} + R_{g,2})(i_a + i_b + i_c) + (v_{b,1} - v_{b,2}), \tag{46}
\]
\[
L \frac{d}{dt} i_c = -R_i c - (R_{g,1} + R_{g,2})(i_a + i_b + i_c) + (v_{c,1} - v_{c,2}),
\]

and direct computations yield
\[
\Lambda_\theta A\Lambda_{\theta}^{-1} = \begin{bmatrix} -\frac{R}{L} & 0 & 0 \\ 0 & -\frac{R}{L} & 0 \\ 0 & 0 & -\frac{R + 3(R_{g,1} + R_{g,2})}{L} \end{bmatrix}. \tag{47}
\]
Based on (38) and assuming $d\theta/dt = \omega_s$ the resulting dq0 model is

$$\begin{align*}
\frac{d}{dt}i_d &= -\frac{R}{L}i_d + \omega_s i_q + \frac{1}{L}(v_{d,1} - v_{d,2}), \\
\frac{d}{dt}i_q &= -\omega_s i_d - \frac{R}{L}i_q + \frac{1}{L}(v_{q,1} - v_{q,2}), \\
\frac{d}{dt}i_0 &= -\frac{1}{L}(R + 3(R_{g,1} + R_{g,2}))i_0 + \frac{1}{L}(v_{0,1} - v_{0,2}).
\end{align*}$$

(48)

Note that in this example $\Lambda g \Lambda \neq A \Lambda g$, but nevertheless the dq0 model is linear and time-invariant.

The expressions for $i_d$, $i_q$ are exactly identical to the ones in (45), and are unaffected by the resistors $R_{g,1}$ and $R_{g,2}$. These resistors affect only the zero component, which according to (1) represents the average current $i_0 = \frac{1}{3}(i_a + i_b + i_c)$. Note that if $R_{g,1} + R_{g,2} \to \infty$ then $i_0 \to 0$, but $i_d$, $i_q$ are unaffected. This example may explain how transformers are used to eliminate undesired average currents in a power system.

**Comparison of phasors and dq0 quantities**

This section discusses the relations between dq0 models and time-varying phasor models. First we will define the link between phasors and dq0 quantities. Consider the balanced three-phase voltage signals

$$\begin{align*}
v_a(t) &= A(t) \cos(\omega_s t + \psi(t)), \\
v_b(t) &= A(t) \cos(\omega_s t + \psi(t) - \frac{2\pi}{3}), \\
v_c(t) &= A(t) \cos(\omega_s t + \psi(t) + \frac{2\pi}{3}).
\end{align*}$$

(49)

Based on the dq0 transformation in (2) with $\theta(t) = \omega_s t$, and using the trigonometric identity $\cos(\alpha + \beta) = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta)$ we have

$$\begin{align*}
\begin{bmatrix} v_d \\ v_q \\ v_0 \end{bmatrix} &= T_{\omega_s t} \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} \\
&= T_{\omega_s t} \begin{bmatrix} A(t) \cos(\omega_s t + \psi(t)) & A(t) \cos(\omega_s t + \psi(t) - \frac{2\pi}{3}) & A(t) \cos(\omega_s t + \psi(t) + \frac{2\pi}{3}) \\ A(t) \cos(\omega_s t + \psi(t) - \frac{2\pi}{3}) & A(t) \cos(\omega_s t + \psi(t)) & A(t) \cos(\omega_s t + \psi(t) + \frac{2\pi}{3}) \\ A(t) \cos(\omega_s t + \psi(t) + \frac{2\pi}{3}) & A(t) \cos(\omega_s t + \psi(t) - \frac{2\pi}{3}) & A(t) \cos(\omega_s t + \psi(t)) \end{bmatrix} \\
&= T_{\omega_s t} \begin{bmatrix} \cos(\omega_s t) & -\sin(\omega_s t) & 1 \\ \cos(\omega_s t - \frac{2\pi}{3}) & -\sin(\omega_s t - \frac{2\pi}{3}) & 1 \\ \cos(\omega_s t + \frac{2\pi}{3}) & -\sin(\omega_s t + \frac{2\pi}{3}) & 1 \end{bmatrix} \begin{bmatrix} A(t) \cos(\psi(t)) \\ A(t) \sin(\psi(t)) \\ 0 \end{bmatrix} \\
&= T_{\omega_s t} T_{\omega_s t}^{-1} \begin{bmatrix} A(t) \cos(\psi(t)) \\ A(t) \sin(\psi(t)) \\ 0 \end{bmatrix} = \begin{bmatrix} A(t) \cos(\psi(t)) \\ A(t) \sin(\psi(t)) \\ 0 \end{bmatrix},
\end{align*}$$

(50)

and therefore

$$\begin{align*}
v_d(t) &= A(t) \cos(\psi(t)), \\
v_q(t) &= A(t) \sin(\psi(t)), \\
v_0(t) &= 0.
\end{align*}$$

(51)

In addition, assuming that variations in the magnitude $A(t)$ and phase $\psi(t)$ are “slow” (see definition below) in comparison to the nominal system frequency $\omega_s$, the voltages above may be represented by the time-varying phasor

$$\begin{align*}
V(t) &= \frac{A(t)}{\sqrt{2}} e^{j\psi(t)}.
\end{align*}$$

(52)
Following (51) and (52) the phasor $V(t)$ may be expressed in terms of $dq0$ quantities as

\[ V(t) = \frac{A(t)}{\sqrt{2}} e^{j\psi(t)} = \frac{A(t)}{\sqrt{2}} (\cos(\psi(t)) + j \sin(\psi(t))) = \frac{1}{\sqrt{2}} \left( v_d(t) + j v_q(t) \right), \]

or alternatively,

\[ v_d(t) = \sqrt{2} \text{Re}\{V(t)\}, \quad v_q(t) = \sqrt{2} \text{Im}\{V(t)\} \quad v_0(t) = 0. \]

Note that since $v_0(t) = 0$ this definition holds only for balanced three-phase signals.

Similarly the active power, reactive power, amplitude and phase may be defined with respect to the $dq$ components as follows:

\begin{align*}
P(t) &= \text{Re}\{V(t)I^*(t)\} = \frac{1}{2} (v_d(t)i_d(t) + v_q(t)i_q(t)), \\
Q(t) &= \text{Im}\{V(t)I^*(t)\} = \frac{1}{2} (v_q(t)i_d(t) - v_d(t)i_q(t)), \\
|V(t)|^2 &= \text{Re}\{V(t)^2\} + \text{Im}\{V(t)^2\} = \frac{1}{2} (v_d^2(t) + v_q^2(t)), \\
\angle V(t) &= \text{atan2}(v_q, v_d).
\end{align*}

We will now examine the similarities and differences between $dq0$ models and time-varying phasor models. Consider the symmetrically configured inductor, which dynamic model is given in (16). Assuming $\theta = \omega_s t$ and $v_0 = 0$ this model may be written as

\begin{align*}
v_d(t) &= -\omega_s L i_q(t) + L \frac{d}{dt} i_d, \\
v_q(t) &= \omega_s L i_d(t) + L \frac{d}{dt} i_q.
\end{align*}

Using time-varying phasors the inductor impedance is $j\omega_s L$, and therefore

\[ I(t) = \frac{1}{j\omega_s L} V(t), \]

where $V(t)$ and $I(t)$ are the phasors representing the inductor voltage and current. The real and imaginary parts of this last equation are

\begin{align*}
\text{Re}\{I(t)\} &= \frac{1}{\omega_s L} \text{Im}\{V(t)\}, \\
\text{Im}\{I(t)\} &= -\frac{1}{\omega_s L} \text{Re}\{V(t)\},
\end{align*}

and based on the relations between $dq$ signals and phasors in (54) an equivalent model is

\begin{align*}
i_d(t) &= \frac{1}{\omega_s L} v_q(t), \\
i_q(t) &= -\frac{1}{\omega_s L} v_d(t),
\end{align*}

or

\begin{align*}
v_d(t) &= -\omega_s L i_q(t), \\
v_q(t) &= \omega_s L i_d(t).
\end{align*}
Direct comparison of (56) and (60) reveals that both models are similar, except for the time derivatives in the \(dq0\) model. These derivatives describe the main difference between the two models. While the \(dq0\) model is general and accurate, the time-varying phasor model is an approximation, which only holds for slow variations. If variations in the \(dq\) components are indeed slow such that

\[
\begin{align*}
\frac{d}{dt} i_d &<< \omega_s |i_q(t)|, \\
\frac{d}{dt} i_q &<< \omega_s |i_d(t)|, 
\end{align*}
\]  

then (56) and (60) are almost identical, and therefore time-varying phasors may be used instead of \(dq0\) quantities. This result is extended in [1] and [2], which formulate the relations between \(dq0\) models and time varying phasor models for general three-phase networks.

Remark (Equilibrium points): for balanced systems, since at equilibrium the time derivatives are equal to zero, the \(dq0\) model is identical to the time-varying phasor model. For this reason equilibrium points may be calculated based on phasors. This is typically done by solving the system’s power flow equations, as described in Lecture 1.

**Power expressions for phasors and for \(dq0\) quantities**

In this section we will briefly recall the power definitions we developed so far, and examine how they relate to each other.

Equation (21) defines the *instantaneous three-phase* power:

\[
p_{3\phi}(t) = v_a(t)i_a(t) + v_b(t)i_b(t) + v_c(t)i_c(t) = \frac{3}{2} (v_a(t)i_a(t) + v_q(t)i_q(t) + 2v_0(t)i_0(t)) .
\]  

In addition, in the context of time-varying phasor models we discussed the *active* power:

\[
P(t) = \frac{1}{T} \int_{t-T/2}^{t+T/2} v_a(\tau)i_a(\tau) d\tau = \frac{1}{T} \int_{t-T/2}^{t+T/2} v_b(\tau)i_b(\tau) d\tau = \frac{1}{T} \int_{t-T/2}^{t+T/2} v_c(\tau)i_c(\tau) d\tau .
\]  

Here \(v_a i_a, v_b i_b, v_c i_c\) are the instantaneous single-phase powers, and \(T\) is the period of the AC signals. The active power is the *average* power over a cycle, for a single phase. We have also seen in (55) that under the approximation of time varying phasors

\[
P(t) = \text{Re} \{V(t)I^*(t)\} = \frac{1}{2} (v_a(t)i_a(t) + v_q(t)i_q(t)) , \]

\[
Q(t) = \text{Im} \{V(t)I^*(t)\} = \frac{1}{2} (v_q(t)i_d(t) - v_d(t)i_q(t)) .
\]

Based on (62) and (64) and assuming that in a balanced power system \(v_0 = 0\) we have

\[
p_{3\phi}(t) = 3P(t) ,
\]  

which means that the *total instantaneous* power is equal to three times the *active* power.

Moreover, the approximation of time-varying phasors dictates that variations in \(P(t)\) during a single cycle are small. Therefore, based on (65), the total instantaneous power \(p_{3\phi}(t)\) must be almost constant during a cycle. This fact is one of the greatest advantages of balanced three-phase systems. While a single-phase system provides alternating power, a balanced three-phase system provides almost constant power. As a result, three-phase devices do not need to store significant energy, and can be made small and efficient. This idea is illustrated in Fig. 7.
Example — modeling a network based on dq0 quantities

Consider the linear three-phase network described in Fig. 8. The network includes an ideal power source which is connected to an infinite bus (a voltage source with constant amplitude and frequency).

It is assumed that

- The network and the sources within it are symmetrically configured;
- the reference angle for the dq0 transformation is \( \theta = \omega_s t \), where \( \omega_s \) is the frequency of the infinite bus.
- The power source is modeled as

\[
\frac{1}{2}(v_{d,1}i_d + v_{q,1}i_q) = P, \\
\frac{1}{2}(v_{q,1}i_d - v_{d,1}i_q) = 0, \\
v_{0,1} = 0.
\]

(66)

- The infinite bus is modeled as

\[
v_{d,2} = \sqrt{2}V_g, \\
v_{q,2} = 0, \\
v_{0,2} = 0.
\]

(67)
**Time-varying phasor model**

Here it is assumed that the network may be described based on time-varying phasors. The voltage of the infinite bus is

\[ V_2 = V_g, \]  

and the power source is characterized by

\[ P_1 = P, \]
\[ Q_1 = 0, \]  

where \( V_g \) and \( P \) are given constants.

We define the inductor current as \( I \), and the voltage on the power source as \( V_1 \) (these are phasors). The system equations are

\[ P = V_1 I^*, \]
\[ V_1 - V_g = (j\omega_s L + R)I, \]  

which lead to

\[ P = (V_g + (j\omega_s L + R)I) I^* \]
\[ = V_g I^* + (j\omega_s L + R)|I|^2. \]  

This last equation may be solved to find the current \( I \).

It can be seen that the time-varying phasor model has no dynamic states, and is characterized by a set of algebraic equations. The solution, if it exists, is constant in time.

**DQ0 model**

We will now develop the \( dq0 \) model of the network above assuming that \( R = 0 \).

According to (16) with \( \theta = \omega_s t \) the inductor model is

\[
\frac{d}{dt} i_d = \omega_s i_q + \frac{1}{L} (v_{d,1} - v_{d,2}), \\
\frac{d}{dt} i_q = -\omega_s i_d + \frac{1}{L} (v_{q,1} - v_{q,2}).
\]  

The power source is modeled as in (66), which may be written as

\[
\begin{bmatrix}
  v_{d,1} \\
  v_{q,1}
\end{bmatrix} = \frac{2P}{i_d^2 + i_q^2} \begin{bmatrix}
  i_d \\
  i_q
\end{bmatrix}.
\]  

Combination of (67), (72) and (73) yields the (nonlinear) state-space model

\[
\frac{d}{dt} i_d = \omega_s i_q + \frac{2P}{L} \frac{i_d}{i_d^2 + i_q^2} - \frac{\sqrt{2}V_g}{L}, \\
\frac{d}{dt} i_q = -\omega_s i_d + \frac{2P}{L} \frac{i_q}{i_d^2 + i_q^2},
\]  

which Jacobian at an equilibrium point \((\overline{i_d}, \overline{i_q})\) is

\[
A = \begin{bmatrix}
-\frac{2P}{L} \frac{i_d^2 - i_q^2}{(i_d^2 + i_q^2)^2} & \omega_s - \frac{4P}{L} \frac{i_d i_q}{(i_d + i_q)^2} \\
-\omega_s - \frac{4P}{L} \frac{i_d i_q}{(i_d + i_q)^2} & \frac{2P}{L} \frac{i_d^2 - i_q^2}{(i_d + i_q)^2}
\end{bmatrix}.
\]
The poles are found by computing the roots of the characteristic polynomial, which is given for a second-order system by

\[ s^2 - \text{Tr}(A)s + \det(A) = 0, \tag{76} \]

where \(\text{Tr}(A)\) is the trace of \(A\), and \(\det(A)\) is the determinant of \(A\). It is easy to verify that in our example \(\text{Tr}(A) = 0\), and therefore the poles are \(s_{1,2} = \pm \sqrt{\det(A)}\). Consider the following two cases:

- If \(\det(A) > 0\) there is a pole in the right half of the complex plane, and the system is unstable.
- If \(\det(A) \leq 0\) there is a complex conjugate pair of poles on the imaginary axis, and additional analysis in simulation reveals that the system is unstable.

Based on these results we can conclude that the \(dq0\) model is unstable.

Can we use the approximation of time-varying phasors in this example? We have seen that

- The time-varying phasor model has no dynamic states, and therefore provides no information regarding the system stability.
- With \(R = 0\) the \(dq0\) model is unstable.

The time-varying phasor model may be used to find the equilibrium point(s). However, in case \(R = 0\), the \(dq0\) model reveals that these equilibrium points are unstable, and do not represent physical solutions.

Work [3] considers the same example with an additional capacitor which is connected in parallel to the power source. This additional capacitor acts as an energy storage device, and may help to stabilize the system.

**Appendix: useful \(dq0\) identities**

Here are several useful identities related to the \(dq0\) transformation.

\[
T_\theta = \frac{2}{3} \begin{bmatrix} \cos(\theta) & \cos(\theta - \frac{2\pi}{3}) & \cos(\theta + \frac{2\pi}{3}) \\ \sin(\theta) & -\sin(\theta - \frac{2\pi}{3}) & \sin(\theta + \frac{2\pi}{3}) \end{bmatrix}, \tag{77}
\]

\[
T_\theta^{-1} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 1 \\ \cos(\theta - \frac{2\pi}{3}) & -\sin(\theta - \frac{2\pi}{3}) & 1 \\ \cos(\theta + \frac{2\pi}{3}) & -\sin(\theta + \frac{2\pi}{3}) & 1 \end{bmatrix}, \tag{78}
\]

\[
T_\theta T_\theta^{-1} = T_\theta^{-1} T_\theta = I_{3\times3}, \tag{79}
\]

\[
(T_\theta^{-1})^T T_\theta^{-1} = \frac{3}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \tag{80}
\]

\[
T_\theta (T_\theta)^T = \frac{2}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \tag{81}
\]

\[
T_{\theta_a} T_{\theta_b}^{-1} = \begin{bmatrix} \cos(\theta_a - \theta_b) & \sin(\theta_a - \theta_b) & 0 \\ -\sin(\theta_a - \theta_b) & \cos(\theta_a - \theta_b) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{82}
\]
\[
\frac{d}{dt} T_\theta^{-1} = -T_\theta^{-1} W, \\
\frac{d}{dt} T_\theta = W T_\theta, \\
\]
with
\[
W = \begin{bmatrix}
0 & \frac{d}{dt} \theta & 0 \\
-\frac{d}{dt} \theta & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

All these identities can be proved by straight-forward algebraic calculations.

References


