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TALLINN UNIVERSITY OF TECHNOLOGY

ISS0023 Intelligent Control Systems

Fractional-order Calculus based Modeling and Control of Dynamic Systems

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Lecture overview

- Mathematical basis of fractional-order calculus;
- Fractional-order calculus in modeling and control:
  - Analysis of fractional models;
  - $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ controllers and their design;
  - Implementations of fractional-order systems and controllers.
- Overview of CACSD tools and examples of practical applications:
  - Introduction to FOMCON toolbox for MATLAB;
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Part One: Mathematical Basis of Fractional-order Calculus

Introduction: Historical facts

- The concept of the differentiation operator $\mathscr{D} = \mathrm{d}/\mathrm{d}\,x$ is a well-known fundamental tool of modern calculus. For a suitable function f the n-th derivative is well defined as

$$\mathscr{D}^n f(x) = \mathrm{d}^{\,n} f(x)/\mathrm{d}x^n, \tag{1}$$

where n is a positive integer.

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- What happens if we extend this concept to a situation, when the order of differentiation is arbitrary, for example, fractional?
- That was the very same question L'Hôpital addressed to Leibniz in a letter in 1695. Since then the concept of fractional calculus has drawn the attention of many famous mathematicians, including Euler, Laplace, Fourier, Liouville, Riemann, Abel.

Fractional derivative of a power function: An approach based on intuition

For the power function $f(x) = x^k$ the fractional derivative can be shown to be

$$\frac{\mathrm{d}^\alpha f(x)}{\mathrm{d}x^\alpha} = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} x^{k-\alpha}. \tag{2}$$

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The function $\Gamma(\cdot)$ above is the Gamma function—the generalization of the factorial function:

$$\Gamma(x) = \int_0^\infty t^{x-1} \mathrm{e}^{-t} \mathrm{d}t, \quad x > 0. \tag{3}$$

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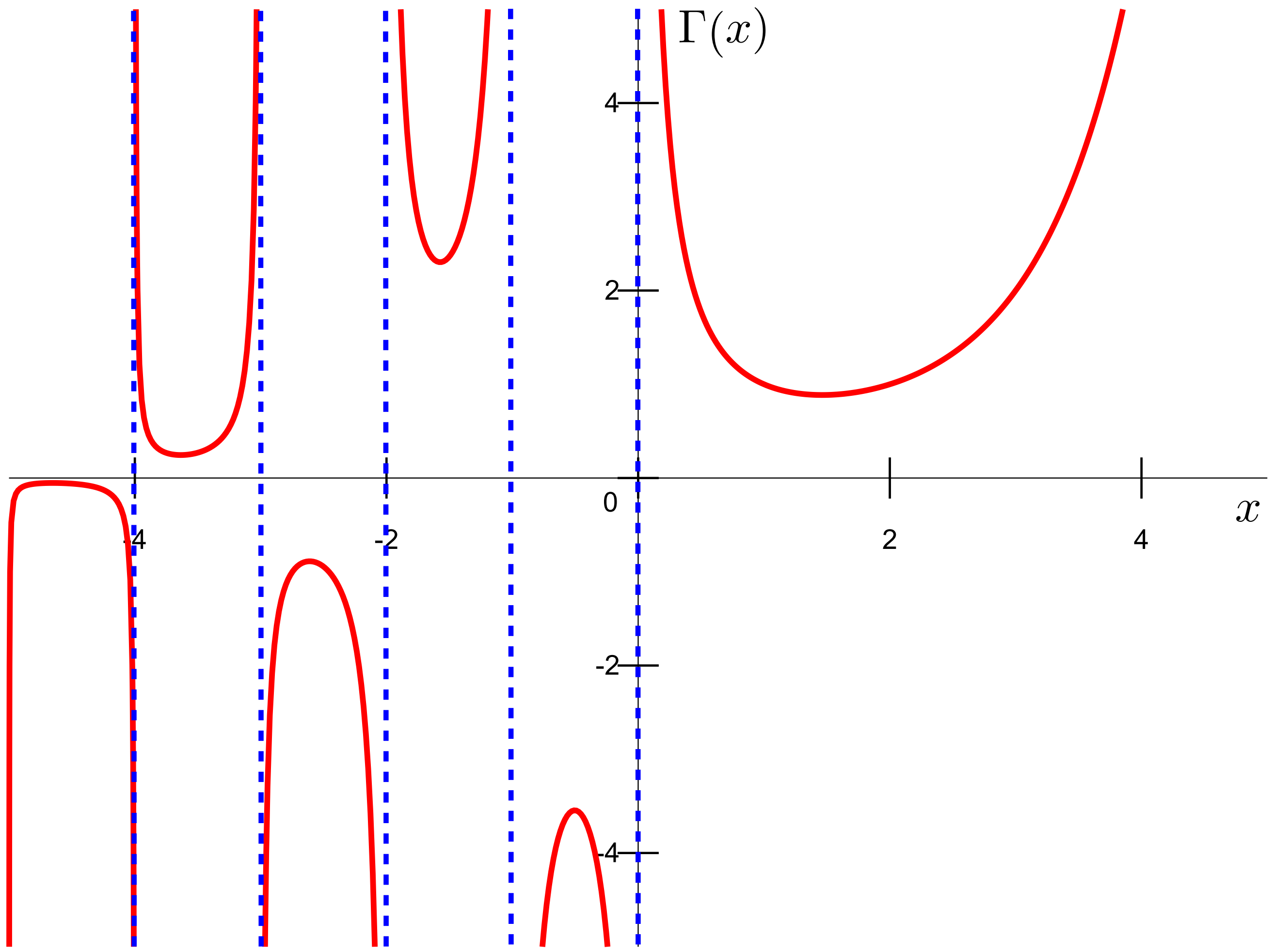
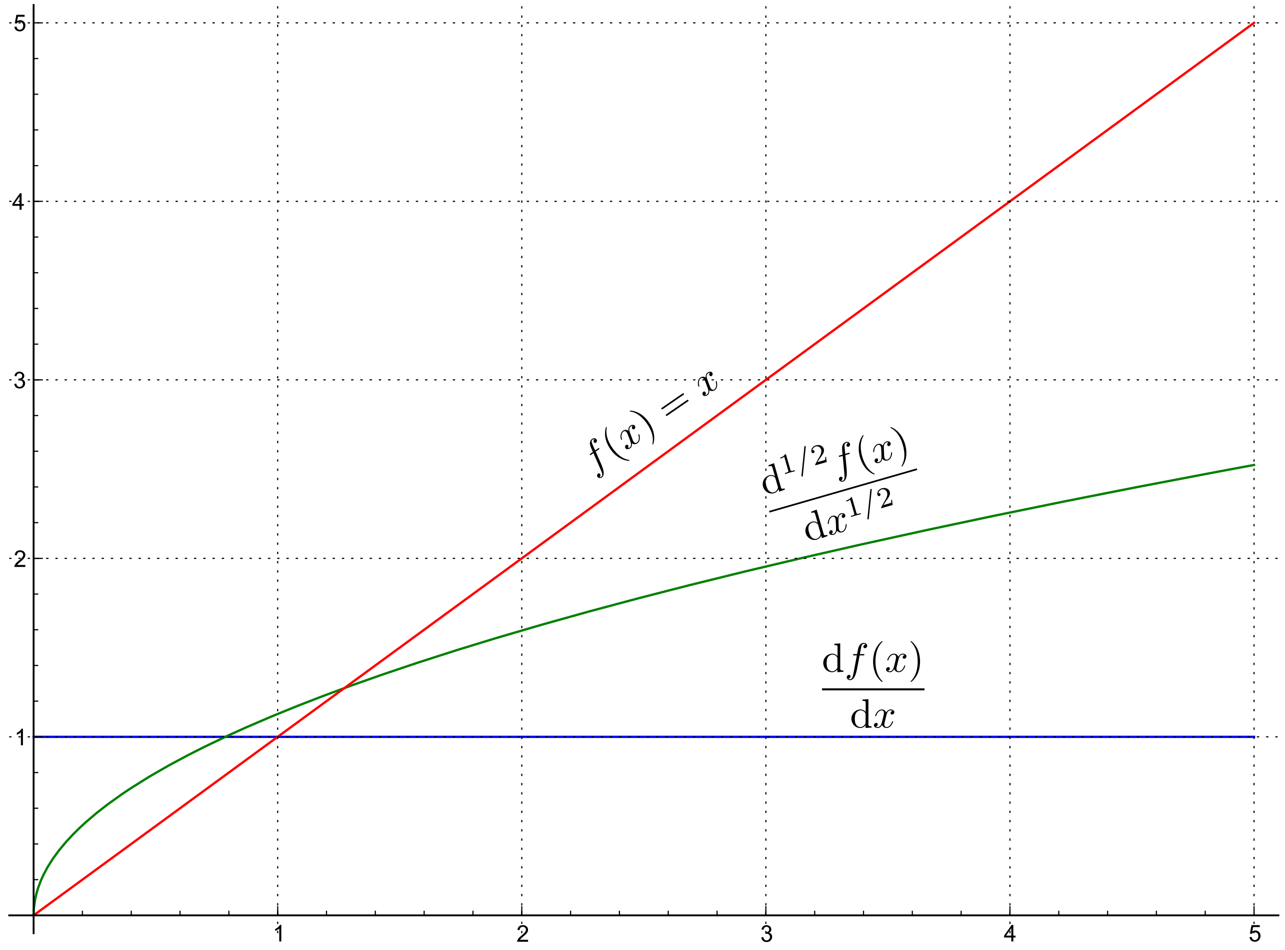
Example:

$$\frac{\mathrm{d}^{1/2}(x^2)}{\mathrm{d}x^{1/2}} = \frac{\Gamma(3)}{\Gamma(5/2)} x^{3/2} = \frac{8x^{3/2}}{3\sqrt{\pi}}.$$

The Gamma function



Example: fractional-order derivative of a function $f(x) = x$



Fractional derivative of a trigonometric function: An approach based on intuition

We observe, what happens when we repeatedly differentiate the function $f(x) = \sin x$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x,\ \frac{\mathrm{d}^2}{\mathrm{d}x^2}\sin x = -\sin x,\ \frac{\mathrm{d}^3}{\mathrm{d}x^3}\sin x = -\cos x,\ \ldots$$

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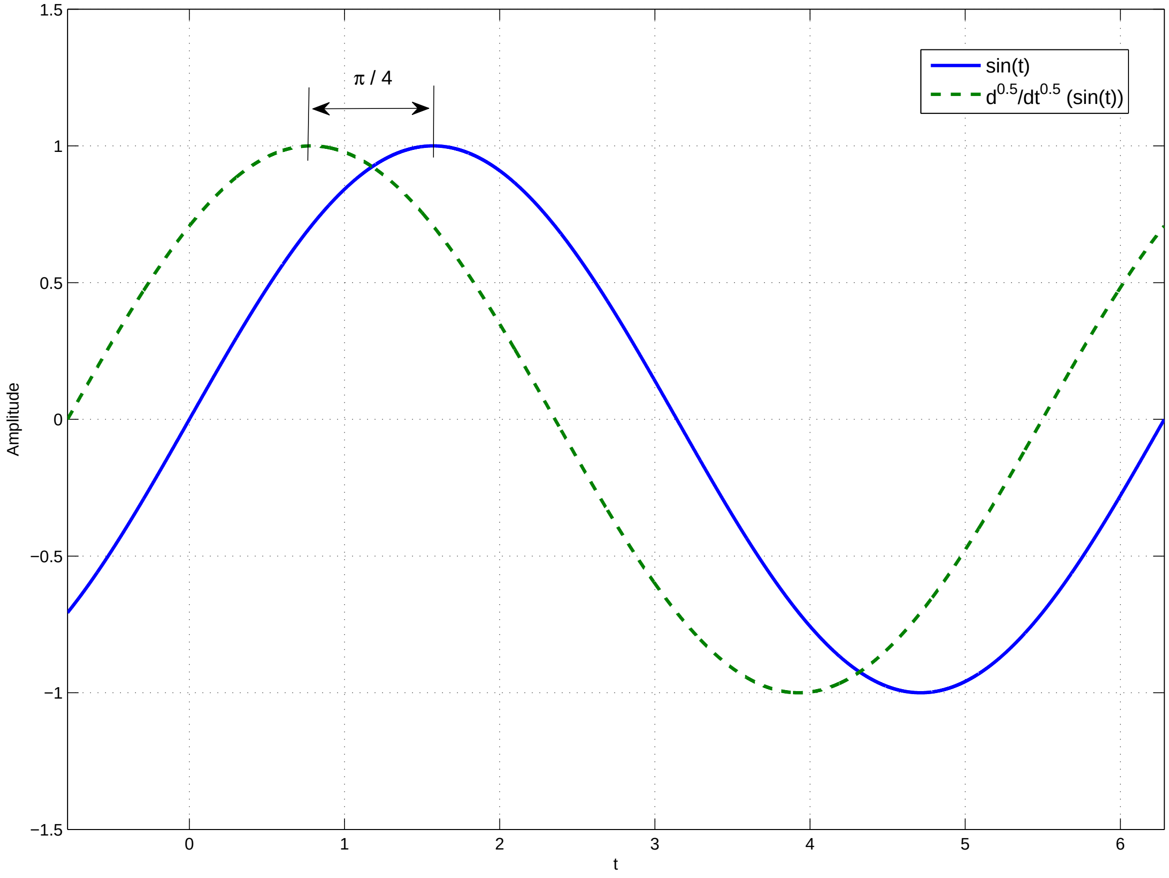
The pattern can be deduced: for the nth derivative, the function $\sin x$ is shifted by $n\pi/2$ radians. This can be observed from studying the graph of the function. Thus, if we replace n by $\alpha \in \mathbb{R}^+$, we have

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}\sin x = \sin\left(x + \frac{\alpha\pi}{2}\right). \tag{4}$$

Obviously, a similar equation holds for the cosine function as well.

Half derivative of a sine function



Repeated differentiation: Backward difference equation

Recall the backward difference definition of $f'(x)$ given by

$$f'(x) = \lim_{h \to 0} \frac{f(x) - f(x-h)}{h}. \tag{5}$$

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And in general

$$f^{(n)}(x) = \lim_{h\to 0} \frac{1}{h^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x-kh). \tag{6}$$

Repeated differentiation: Backward difference equation based generalization

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Of course! All we need to do is to consider the factorial formula for the binomial coefficient and use the ever so kind Gamma function to lend a helping hand in case we have $\alpha \in \mathbb{R}^+$:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \rightarrow \quad \binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}. \tag{7}$$

We find that this approach is the very basis for Grünwald-Letnikov's definition of the fractional-order derivative.

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We find that this approach is the very basis for Grünwald-Letnikov's definition of the fractional-order derivative. In fact, here it is:

Definition 1. *(Grünwald-Letnikov)*

$${}^{GL}\mathscr{D}^{\alpha} f(t)|_{t=nh} = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} f(nh-kh). \tag{8}$$

Fractional-order derivative: Important alternative definitions

Definition 2. *(Riemann-Liouville)*

$$ {}_a^R\mathscr{D}_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m \left[\int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-m+1}}\,\mathrm{d}\tau\right], \qquad (9) $$

where $m-1<\alpha<m$, $m\in\mathbb{N}$,$\alpha\in\mathbb{R}^+$.

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Definition 3. *(Caputo)*

$$ {}_{0}^{C}\mathscr{D}_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}}\mathrm{d}\tau, \qquad (10) $$

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The generalized operator

Fractional calculus is a generalization of integration and differentiation to non-integer order operator ${}_a\mathscr{D}_t^\alpha$, where a and t denote the limits of the operation and α denotes the fractional order such that

$$
{}_a\mathscr{D}_t^\alpha = \begin{cases} \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} & \Re(\alpha) > 0, \\ 1 & \Re(\alpha) = 0, \\ \int_a^t (\mathrm{d}\tau)^{-\alpha} & \Re(\alpha) < 0, \end{cases} \tag{11}
$$

where generally it is assumed that $\alpha \in \mathbb{R}$, but it may also be a complex number. We restrict our attention to the former case.

Properties of fractional-order differentiation

Fractional-order differentiation has the following properties:

1. If $\alpha = n$ and $n \in \mathbb{Z}^+$, then the operator ${}_0\mathscr{D}_t^\alpha$ can be understood as the usual operator $\mathrm{d}^n/\mathrm{d}t^n$.
2. Operator of order $\alpha = 0$ is the identity operator: ${}_0\mathscr{D}_t^0 f(t) = f(t)$.
3. Fractional-order differentiation is linear; if a, b are constants, then

$$ {}_0\mathscr{D}_t^\alpha \left[af(t) + bg(t) \right] = a\,{}_0\mathscr{D}_t^\alpha f(t) + b\,{}_0\mathscr{D}_t^\alpha g(t). \tag{12} $$

4. If $f(t)$ is an analytic function, then the fractional-order differentiation ${}_0\mathscr{D}_t^\alpha f(t)$ is also analytic with respect to t.

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Properties of fractional-order differentiation (continued)

5. For the fractional-order operators with $\Re(\alpha) > 0, \Re(\beta) > 0$, and under reasonable constraints on the function $f(t)$ it holds the additive law of exponents:

$$ {}_0\mathscr{D}_t^{\alpha} \left[{}_0\mathscr{D}_t^{\beta} f(t) \right] = {}_0\mathscr{D}_t^{\beta} \left[{}_0\mathscr{D}_t^{\alpha} f(t) \right] = {}_0\mathscr{D}_t^{\alpha+\beta} f(t) \qquad (13) $$

6. The fractional-order derivative commutes with integer-order derivative

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and if $t = a$ we have $f^{(k)}(a) = 0,\ (k = 0,\ 1,\ 2,\ \ldots,\ n-1)$.

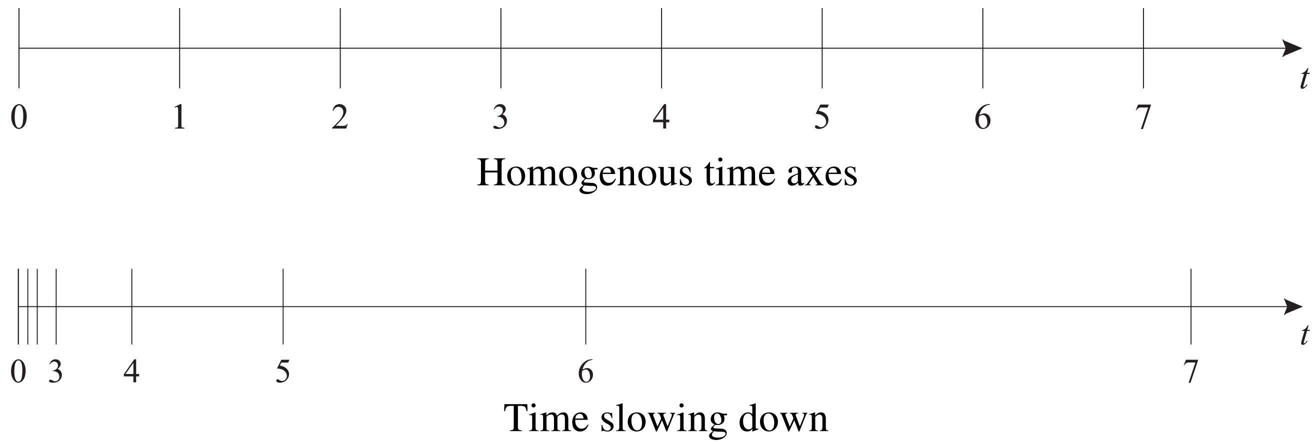
On the meaning of the fractional-order derivative

Consider a hereditary operator $\mathscr{F}(f_t(\cdot), t)$ acting on a cause process $f_t(\cdot)$ to produce a time-shifted effect $g(t)$ which depends on the history of the process $\{f_t(\tau); \tau < t\}$:

$$g(t) = \mathscr{F}\left[f_t(\cdot); t\right]. \tag{15}$$

The main idea: The fractional-order operator is *non-local*. Hereditary process examples from physics: Brownian motion; Viscoelasticity.

Suggested relation to time scales [5]:



Exercise: Integration

Compute a fractional-order derivative of order $1/2$ for the function $f(t) = t^2$ using the Caputo definition. Hint: $\Gamma(1/2) = \sqrt{\pi}$.

$$ {}_0^C\mathscr{D}_t^{1/2} t^2 = \frac{1}{\Gamma(1 - 1/2)} \int_0^t \frac{(\tau^2)'}{(t-\tau)^{1/2-1+1}} \mathrm{d}\tau = \quad ? $$

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The answer is

$$ \frac{1}{\sqrt{\pi}} \cdot \left(4/3(t-\tau)^{3/2} - 4t\sqrt{t-\tau} + C \right)\Big|_0^t $$

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$$ \tfrac{1}{\sqrt{\pi}} \cdot \left({}^4\!/_3 (t-\tau)^{3/2} - 4t\sqrt{t-\tau} + C \right)\Big|_0^t = \tfrac{1}{\sqrt{\pi}} \cdot \left(-{}^4\!/_3 t^{3/2} + 4t^{3/2} \right) $$

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$$ \frac{1}{\sqrt{\pi}} \cdot \left({}^4\!/_3 (t-\tau)^{3/2} - 4t\sqrt{t-\tau} + C \right) \Bigg|_0^t = \frac{1}{\sqrt{\pi}} \cdot \left(-{}^4\!/_3 t^{3/2} + 4t^{3/2} \right) = \frac{8t^{3/2}}{3\sqrt{\pi}}. $$

Part Two: Factional-order Calculus in Modeling and Control of Dynamic Systems

Laplace transform

A function $F(s)$ of the complex variable s is called the Laplace transform of the original function $f(t)$ and defined as

$$F(s) = \mathscr{L}\left[f(t)\right] = \int_0^\infty \mathrm{e}^{-st} f(t)\mathrm{d}t \tag{16}$$

The original function $f(t)$ can be recovered from the Laplace transform $F(s)$ by applying the inverse Laplace transform

$$f(t) = \mathscr{L}^{-1}\left[F(s)\right] = \frac{1}{j2\pi}\int_{c-j\infty}^{c+j\infty} \mathrm{e}^{st} F(s)\mathrm{d}s, \tag{17}$$

where c is greater than the real part of all the poles of $F(s)$.

Fractional-order derivative definitions: Laplace transform

Definition 4. *(Riemann-Liouville)*

$$\mathscr{L}\left[{}^{R}\mathscr{D}^{\alpha} f(t)\right] = s^{\alpha}F(s) - \sum_{k=0}^{m-1} s^{k}\left[\mathscr{D}^{\alpha-k-1} f(t)\right]_{t=0}. \qquad (18)$$

Definition 5. *(Caputo)*

$$\mathscr{L}\left[{}^{C}\mathscr{D}^{\alpha} f(t)\right] = s^{\alpha}F(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(0). \qquad (19)$$

Definition 6. *(Grünwald-Letnikov)*

$$\mathscr{L}\left[{}^{L}\mathscr{D}^{\alpha} f(t)\right] = s^{\alpha}F(s). \qquad (20)$$

For the first two definitions we have $(m-1 \leqslant \alpha < m)$.

Fractional-order models

A linear, fractional-order continuous-time dynamic system can be expressed by a fractional differential equation of the following form

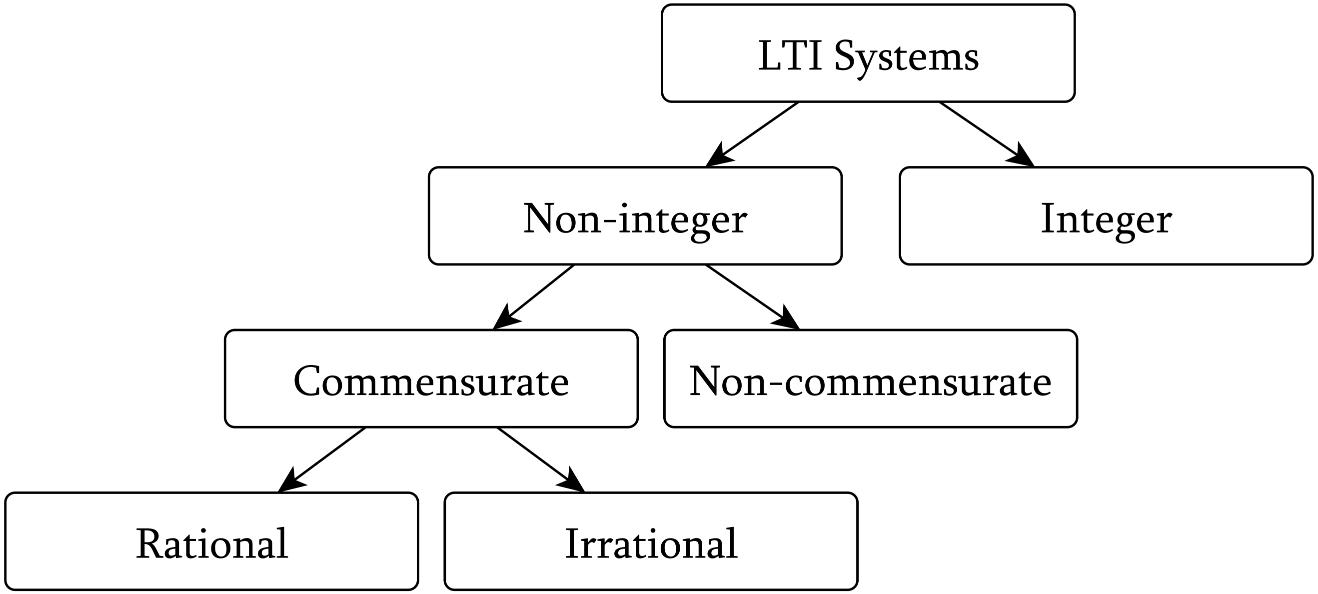
$$\begin{aligned} a_n \mathscr{D}^{\alpha_n} y(t) + a_{n-1} \mathscr{D}^{\alpha_{n-1}} y(t) + \cdots + a_0 \mathscr{D}^{\alpha_0} y(t) &= \\ b_m \mathscr{D}^{\beta_m} u(t) + b_{m-1} \mathscr{D}^{\beta_{m-1}} u(t) + \cdots + b_0 \mathscr{D}^{\beta_0} u(t), & \end{aligned} \tag{21}$$

where $a_k,\ b_k \in \mathbb{R}$. The system is said to be of *commensurate-order* if in (21) all the orders of derivation are integer multiples of a base order γ such that $\alpha_k,\ \beta_k = k\gamma,\ \gamma \in \mathbb{R}^+$. The system can then be expressed as

$$\sum_{k=0}^{n} a_k \mathscr{D}^{k\gamma} y(t) = \sum_{k=0}^{m} b_k \mathscr{D}^{k\gamma} u(t). \tag{22}$$

Linear, time invariant fractional-order system classification

If in (22) the order is $\gamma = 1/q$, $q \in \mathbb{Z}^+$, the system will be of rational order. The diagram with linear time-invariant (LTI) system classification is given in the following diagram.



Fractional-order transfer functions

Applying the Laplace transform to (21) with zero initial conditions the input-output representation of the fractional-order system can be obtained in the form of a transfer function:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^{\beta_m} + b_{m-1} s^{\beta_{m-1}} + \cdots + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \cdots + a_0 s^{\alpha_0}}. \tag{23}$$

In the case of a system with commensurate order γ we have

$$G(s) = \frac{\sum\limits_{k=0}^{m} b_k \left(s^\gamma\right)^k}{\sum\limits_{k=0}^{n} a_k \left(s^\gamma\right)^k}. \tag{24}$$

Fractional-order transfer functions and state-space representation

Taking $\lambda = s^\gamma$ the function (24) can be viewed as a pseudo-rational function $H(\lambda)$:

$$H(\lambda) = \frac{\sum\limits_{k=0}^{m} b_k \lambda^k}{\sum\limits_{k=0}^{n} a_k \lambda^k}. \tag{25}$$

Based on the concept of the pseudo-rational function, a state-space representation can be established in the form:

$$\begin{aligned} \mathscr{D}^\gamma x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t). \end{aligned} \tag{26}$$

Example: From a FO transfer function to the FO state-space form

Suppose that we are given a fractional-order transfer function

$$G(s) = \frac{s^{0.25} + 2.5}{3s^{1.75} + 2s^{0.5} + 1}.$$

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$$G(s) = \frac{s^{0.25} + 2.5}{3s^{1.75} + 2s^{0.5} + 1}.$$

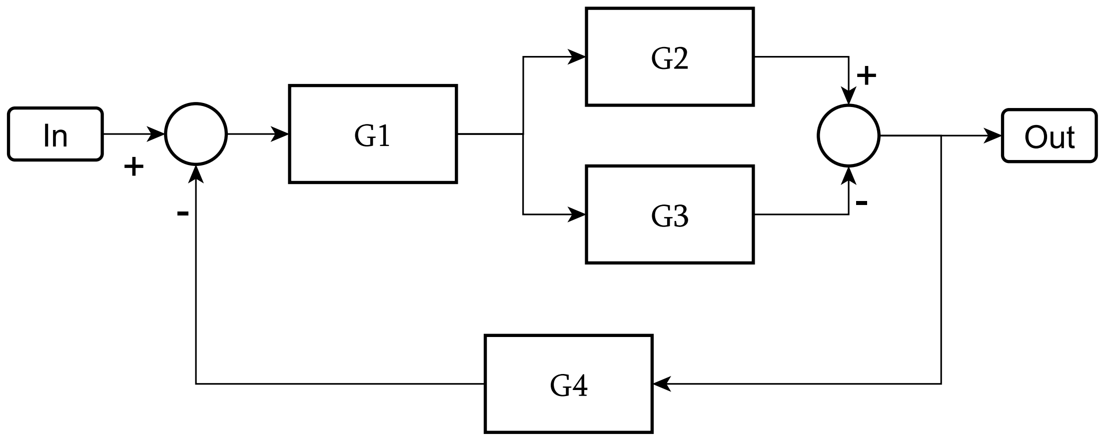
We find, that the commensurate order for this system is $\gamma = 0.25$. Then we use $H(s) = C(sI - A)^{-1}B + D$ and arrive at the following state-space matrices

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & -0.66 & 0 & -0.33 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0.33 & 0.83 \end{bmatrix}, \quad D = 0.$$

Example: Fractional system composition

Let us assume that a fractional system is given by a block diagram



Here

$$
\begin{aligned}
G_1(s) &= \frac{1}{s^{0.5}+1}, \quad G_2(s) = \frac{s^{0.3}+1}{s^{2.5}+s+1},\\
G_3(s) &= \frac{2}{s^{0.1}+1}, \quad G_4(s) = \frac{1}{15s+1}.
\end{aligned}
$$

Compute the transfer function resulting from the interconnection above.

Example: Fractional system composition (solution)

The fractional-order systems we consider are linear. Therefore, the usual operations for computing system interconnections hold. In this case the complete system is given by

$$G(s) = \frac{\begin{array}{c}-30s^{3.5} - 2s^{2.5} - 30s^{2} + 15s^{1.4} + 15s^{1.3} \\ +15s^{1.1} - 17s + s^{0.4} + s^{0.3} + s^{0.1} - 1\end{array}}{\begin{array}{c}15s^{4.1} + 15s^{4} + 15s^{3.6} + 15s^{3.5} + s^{3.1} + s^{3} + 16s^{2.6} \\ +14s^{2.5} + 15s^{2.1} + 15s^{2} + 16s^{1.6} + 16s^{1.5} \\ +16s^{1.1} + 14s + s^{0.6} + s^{0.5} + s^{0.4} + s^{0.3} + 2s^{0.1}\end{array}}.$$

It can be seen from this example that from relatively simple initial systems a fairly complicated fractional-order transfer function was obtained. In this case we find, that the commensurate order of the system is $\gamma = 0.1$.

Stability

Theorem 1. *(Matignon's stability theorem) The fractional transfer function $G(s) = Z(s)/P(s)$ is stable if and only if the following condition is satisfied in σ-plane:*

$$\left|\arg(\sigma)\right| > q\frac{\pi}{2}, \forall\sigma \in \mathbb{C},\ P(\sigma) = 0, \tag{27}$$

where $\sigma := s^q$. When $\sigma = 0$ is a single root of $P(s)$, the system cannot be stable. For $q = 1$, this is the classical theorem of pole location in the complex plane: no pole is in the closed right plane of the first Riemann sheet.

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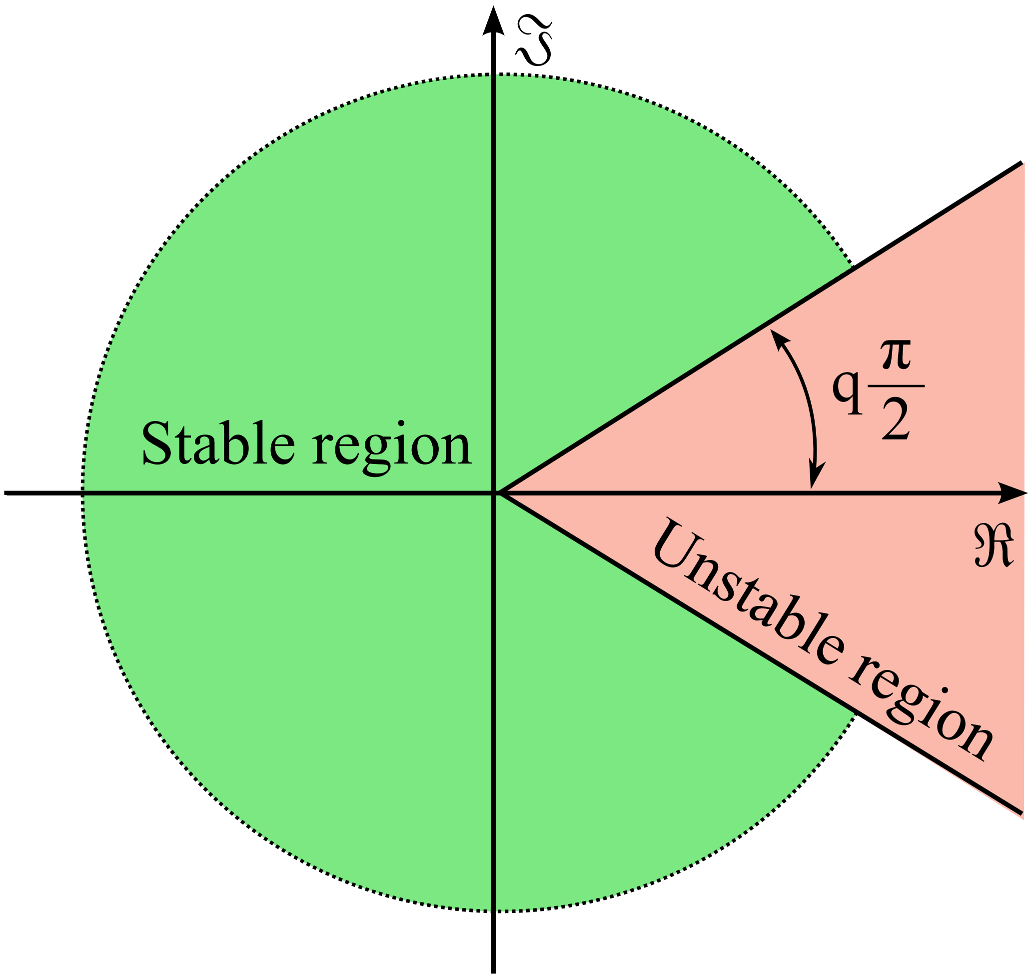
Algorithm summary: Find the commensurate order q of $P(s)$, find $a_1, a_2, \ldots a_n$ in (24) and solve for σ the equation $\sum_{k=0}^{n} a_k \sigma^k = 0$. If all obtained roots satisfy the condition (27), the system is stable.

Stability regions



Exercise: Stability

Determine the commensurate order γ of the fractional-order system given below. Then, write out and solve the characteristic equation $P(\lambda) = 0$. Hint: $\lambda = s^{\gamma}$.

$$G(s) = \frac{s+1}{s - 2s^{0.5} + 5}.$$

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Solution: The commensurate order is $\gamma = 0.5$, so we have $\lambda = s^{0.5}$. Therefore, the characteristic equation is

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Solving $P(\lambda) = 0$ yeilds complex roots $\lambda_{1,2} = 1 \pm j2$. Notice, that in case of a classical integer-order system this result would immediately imply instability. However, in case of this system we have

$$|\arg(1 \pm j2)| \approx 1.1071 > 0.7854 \approx \frac{0.5\pi}{2},$$

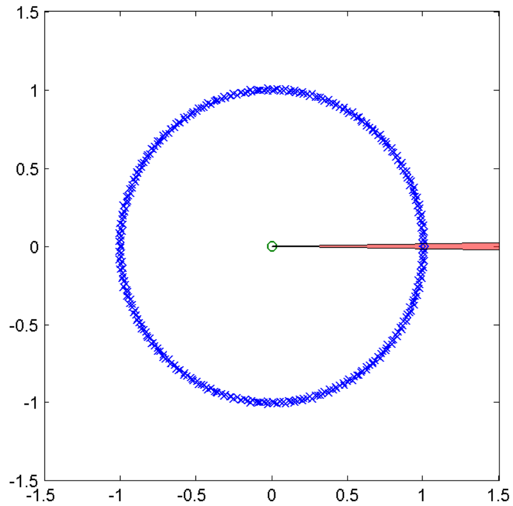
hence the system under analysis is stable.

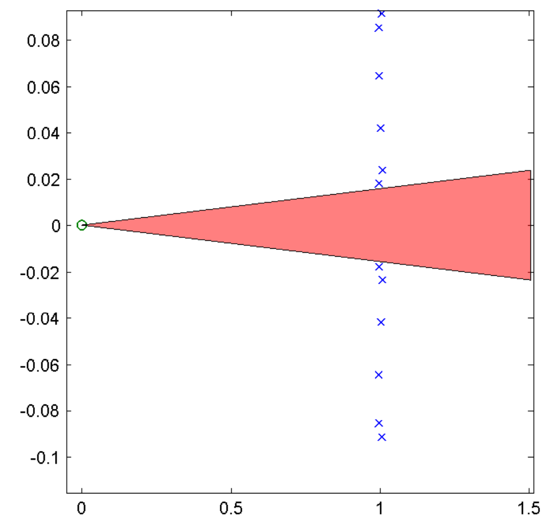
Example: Stability evaluation of a relatively complex system

The transfer function is

$$G(s) = \frac{-2s^{0.63} + 4}{2s^{3.501} + 3.8s^{2.42} + 2.6s^{1.798} + 2.5s^{1.31} + 1.5}$$

and the commensurate order $q = 0.01$. It is found to be stable.





Time-domain analysis

Consider a revised Grünwald-Letnikov definition rewritten as

$$ {}_a\mathscr{D}_t^{\alpha} f(t) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{j=0}^{\left[\frac{t-a}{h}\right]} w_j^{(\alpha)} f(t - jh), \tag{28} $$

where h is the computation step-size and $w_j^{(\alpha)} = (-1)^j \binom{\alpha}{j}$ can be evaluated recursively from

$$ w_0^{(\alpha)} = 1,\ w_j^{(\alpha)} = \left(1 - \frac{\alpha+1}{j}\right) w_{j-1}^{(\alpha)},\ j = 1,\ 2,\ \cdots . \tag{29} $$

Further manipulations provide an algorithm for fixed-step numerical time-domain evaluation of fractional-order transfer functions. Please see [3] for details.

Frequency-domain analysis

Frequency-domain response may be obtained by substituting $s = j\omega$ in (23). The complex response for a frequency $\omega \in (0; \infty)$ can then be computed as follows:

$$G(\omega) = \frac{b_m(j\omega)^{\beta_m} + b_{m-1}(j\omega)^{\beta_{m-1}} + \cdots + b_0(j\omega)^{\beta_0}}{a_n(j\omega)^{\alpha_n} + a_{n-1}(j\omega)^{\alpha_{n-1}} + \cdots + a_0(j\omega)^{\alpha_0}}, \tag{30}$$

where j is the imaginary unit.

It should be noted, that frequency-domain analysis is a very important tool where fractional-order modeling and control design are concerned.

Time-domain identification: Output error minimization

Given the transfer function model in (23)

$$G(s) = \frac{b_m s^{\beta_m} + b_{m-1} s^{\beta_{m-1}} + \cdots + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \cdots + a_0 s^{\alpha_0}}$$

we search for a parameter set $\theta = [\begin{array}{cccc} a_p & \alpha_p & b_z & \beta_z \end{array}]$, such that

$$\begin{aligned} a_p &= [\begin{array}{cccc} a_n & a_{n-1} & \cdots & a_0 \end{array}],\ \alpha_p = [\begin{array}{cccc} \alpha_n & \alpha_{n-1} & \cdots & \alpha_0 \end{array}], \\ b_z &= [\begin{array}{cccc} b_m & b_{m-1} & \cdots & b_0 \end{array}],\ \beta_z = [\begin{array}{cccc} \beta_n & \beta_{n-1} & \cdots & \beta_0 \end{array}], \end{aligned}$$

by employing numerical optimization with an objective function given by an output error norm $\|e\,(t)\|_2^2$, where $e(t) = y(t) - \tilde{y}(t)$ is obtained by taking the difference of the original model output $y(t)$ and simulated model output $\tilde{y}(t)$.

Time-domain identification: Process models

Consider the following generalizations of conventional process models used in industrial control design.

(FO)FOPDT	$G(s) = \frac{K}{1+Ts}\mathrm{e}^{-Ls}$	$G(s) = \frac{K}{1+Ts^{\alpha}}\mathrm{e}^{-Ls}$
(FO)IPDT	$G(s) = \frac{K}{s}\mathrm{e}^{-Ls}$	$G(s) = \frac{K}{s^{\alpha}}\mathrm{e}^{-Ls}$
(FO)FOIPDT	$G(s) = \frac{K}{s(1+Ts)}\mathrm{e}^{-Ls}$	$G(s) = \frac{K}{s(1+Ts^{\alpha})}\mathrm{e}^{-Ls}$

Therefore, due to additional parameters K (gain) and L (delay) we may update the identified parameter set discussed previously to

$$\theta = \left[\begin{array}{cccccc} K & L & a_p & \alpha_p & b_z & \beta_z \end{array} \right].$$

Fractional-order Control: PI$^\lambda$D$^\mu$ controller

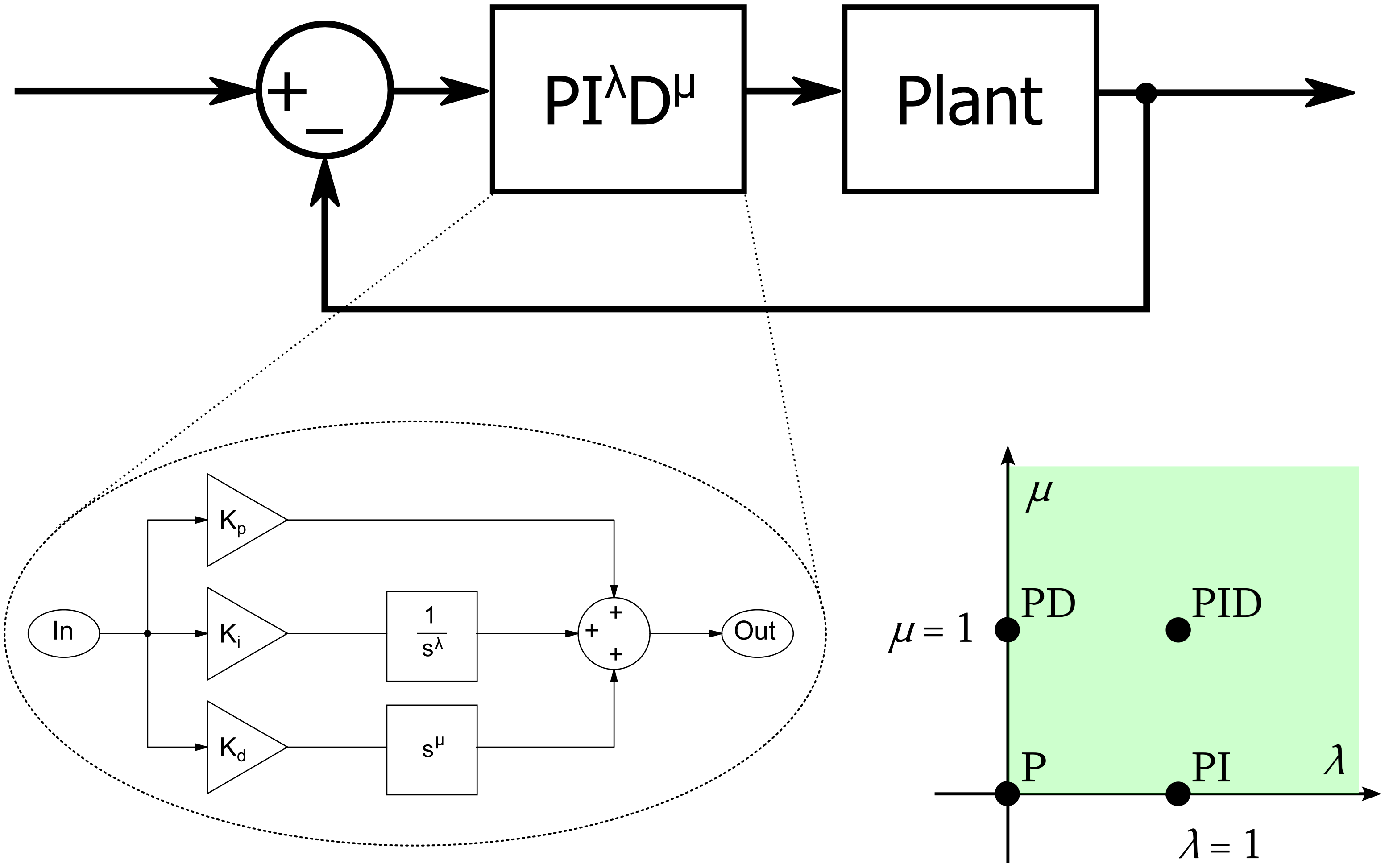
The control law of the PI$^\lambda$D$^\mu$ controller can be expressed as follows:

$$u(t) = K_p e(t) + K_i \mathscr{D}^{-\lambda} e(t) + K_d \mathscr{D}^{\mu} e(t), \tag{31}$$

where $e(t) = y_{sp}(t) - y(t)$ is the error signal. After applying the Laplace transform to (31) assuming zero initial conditions, the following equation is obtained:

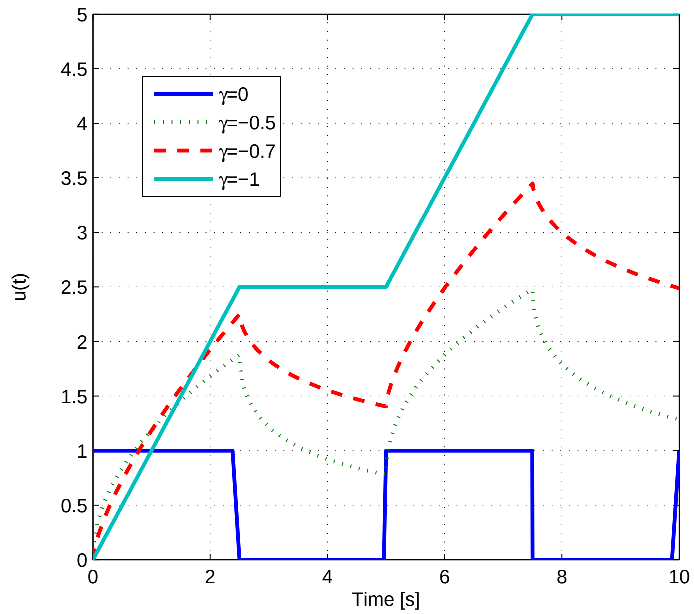
$$C(s) = K_p + \frac{K_i}{s^\lambda} + K_d s^\mu \tag{32}$$

Obviously, when taking $\lambda = \mu = 1$ the result is the classical integer-order PID controller.

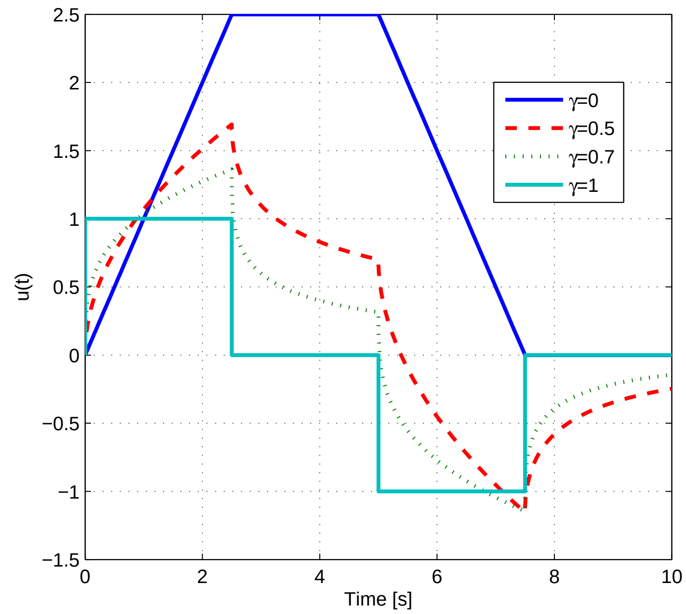

+
_
PIλDμ
Plant
In
Kp
Ki
Kd
1
sλ
sμ
+
+
+
Out
μ
PD
PID
μ = 1
P
PI
λ
λ = 1

Basics of fractional control: Fractional control actions

Let a basic fractional control action be defined as $C(s) = K \cdot s^{\gamma}$. The control actions in the time domain for $\gamma \in [-1, 1]$ with $K = 1$ under different input signals are given below.

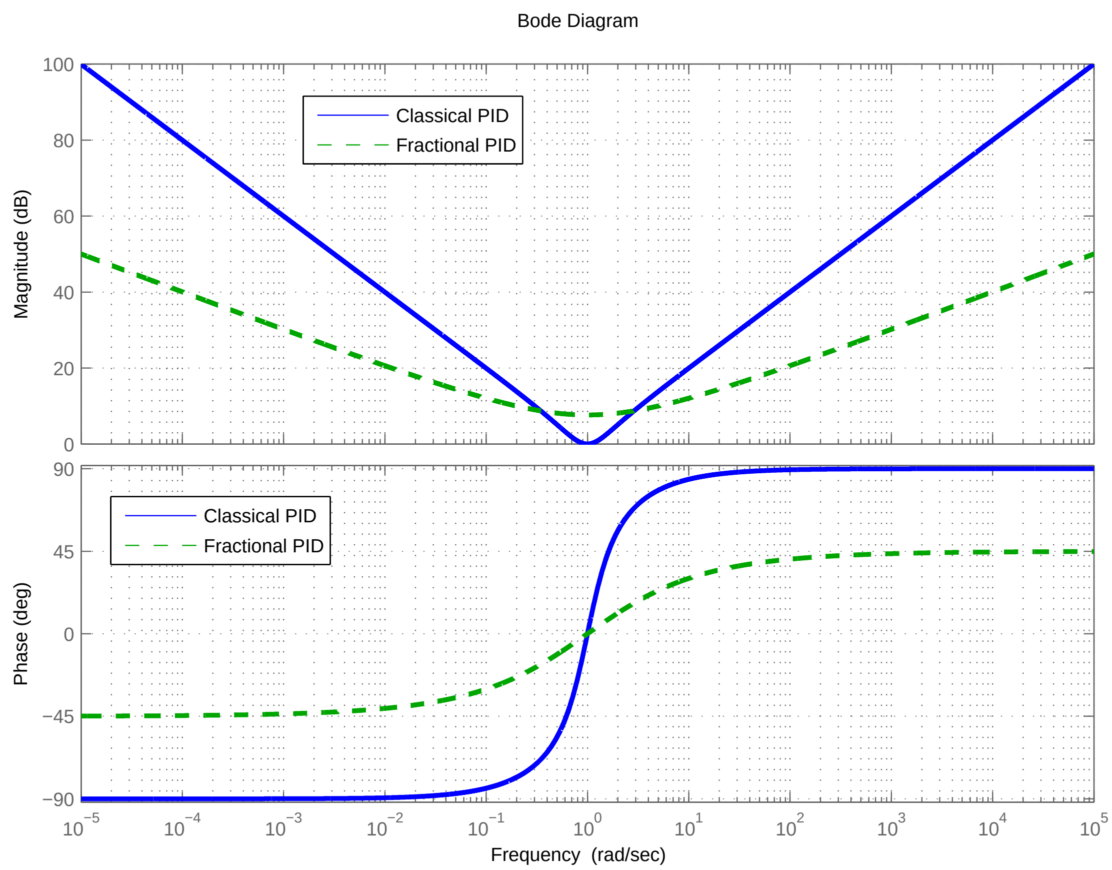


Fractional integrator $s^{-\gamma}$



Fractional differentiator s^{γ}

PID controller vs. $PI^{0.5}D^{0.5}$ controller: frequency-domain characteristics



Fractional-order Control: PI$^\lambda$D$^\mu$ controller tuning methods (F-MIGO)

We would like to establish tuning methods for the FOPID controller similar to conventional ones (e.g. Ziegler-Nichols tuning formulae). Several methods have been proposed in literature so far. Consider the F-MIGO method suitable for tuning PI$^\lambda$ controllers [3]. Suppose we are given a FOPDT process model

$$G(s) = \frac{K}{Ts+1}\mathrm{e}^{-Ls}, \quad \tau = \frac{L}{L+T}, \tag{33}$$

where τ is the relative dead-time of the system.

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where τ is the relative dead-time of the system.Then

$$\lambda = \begin{cases} 1.1, & \text{if } \tau \geqslant 0.6 \\ 1.0, & \text{if } 0.4 \leqslant \tau < 0.6 \\ 0.9, & \text{if } 0.1 \leqslant \tau < 0.4 \\ 0.7, & \text{if } \tau < 0.1. \end{cases}$$

and

$$K_p = \frac{1}{K}\left(\frac{0.2978}{\tau + 0.000307}\right), \quad K_i = \frac{K_p(\tau^2 - 3.402\tau + 2.405)}{0.8578T}.$$

Optimization based PI$^\lambda$D$^\mu$ tuning

Optimization provides general means of tuning a fractional-order PID controller given a cost function and suitable optimization constraints. There are several aspects to the problem of designing a proper controller using constrained optimization:

- The type of plant to be controlled (integer or noninteger order, nonlinear);
- Optimization criterion (cost function);
- Fractional controller design specifications;
- Specific parameters to optimize in the set $\{K_p, K_i, K_d, \lambda, \mu\}$;
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Optimization based PI$^\lambda$D$^\mu$ tuning: Cost function

In case of a linear model we use time-domain simulation of a typical negative unity feedback loop

$$G_{cs}(s) = \frac{C(s)G(s)}{1 + C(s)G(s)}. \tag{34}$$

For the cost function we consider performance indicies:

- integral square error $ISE = \int_0^\tau e^2(t)\mathrm{d}t$,
- integral absolute error $IAE = \int_0^\tau \left|e(t)\right| \mathrm{d}t$,
- integral time-square error $ITSE = \int_0^\tau te^2(t)\mathrm{d}t$,
- integral time-absolute error $ITAE = \int_0^\tau t\left|e(t)\right| \mathrm{d}t$.

Optimization based PI$^{\lambda}$D$^{\mu}$ tuning: Constraints

The design specifications include:

- Gain margin G_m and phase margin φ_m specifications;
- Complementary sensitivity function $T(j\omega)$ constraint, providing A dB of noise attenuation for frequencies $\omega > \omega_t$ rad/s;
- Sensitivity function $S(j\omega)$ constraint for output disturbance rejection, providing a sensitivity function of B dB for frequencies $\omega < \omega_s$ rad/s;
- Robustness to plant gain variations: a flat phase of the system is desired within a region of the system critical frequency ω_{cg};
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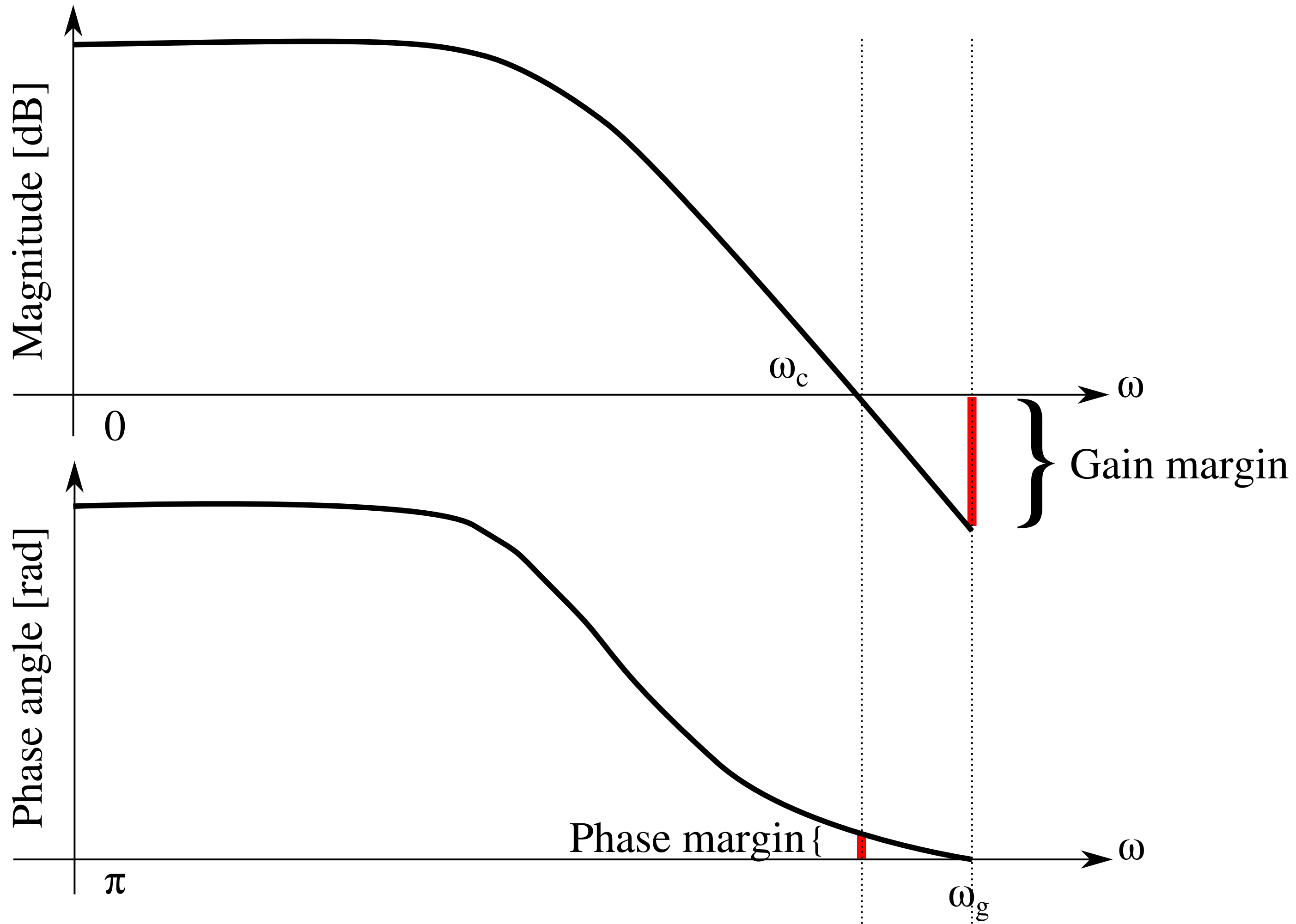
Gain and phase margin specifications



Approximation of fractional operators

The Oustaloup recursive filter gives a very good approximation of fractional operators in a specified frequency range and is widely used in fractional calculus. For a frequency range (ω_b, ω_h) and of order N the filter for an operator s^γ, $0 < \gamma < 1$, is given by

$$
\begin{aligned}
s^\gamma &\approx K \prod_{k=-N}^{N} \frac{s+\omega_k'}{s+\omega_k}, \quad K = \omega_h^\gamma, \quad \omega_r = \frac{\omega_h}{\omega_b}, \\
\omega_k' &= \omega_b (\omega_r)^{\frac{k+N+\frac{1}{2}(1-\gamma)}{2N+1}}, \quad \omega_k = \omega_b (\omega_r)^{\frac{k+N+\frac{1}{2}(1+\gamma)}{2N+1}}.
\end{aligned} \tag{35}
$$

The resulting model order is $2N+1$.

A modified Oustaloup filter has been proposed in literature [3].

Approximation of fractional-order models

A general method for approximating a fractional-order model by an integer-order one may be proposed. Recall the property in (14):

- The fractional-order derivative commutes with integer-order derivative

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left({}_a\mathscr{D}_t^{\alpha} f(t)\right) = {}_a\mathscr{D}_t^{\alpha}\left(\frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}\right) = {}_a\mathscr{D}_t^{\alpha+n} f(t).$$

Thus, for fractional orders $\alpha \geq 1$ it holds

$$s^{\alpha} = s^n s^{\gamma}, \tag{36}$$

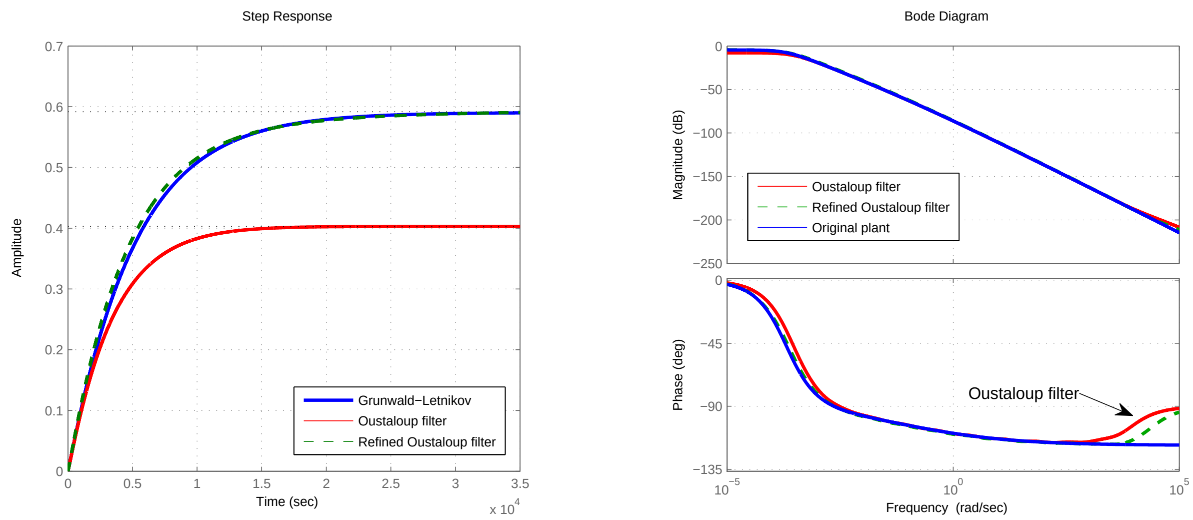
where $n = \alpha - \gamma$ denotes the integer part of α and s^{γ} is obtained by the Oustaloup approximation in (35).

Example: Oustaloup filter approximation

The fractional-order transfer function is

$$G(s) = \frac{1}{14994s^{1.31} + 6009.5s^{0.97} + 1.69},$$

and approximation parameters $\omega = [10^{-4}; 10^{4}]$, $N = 5$.



Discrete-time approximation: The zero-pole matching equivalents method

Continuous zeros and poles, obtained using the Oustaloup recursive filter, are directly mapped to their discrete-time counterparts by means of the relation

$$z = \mathrm{e}^{sT_s}, \tag{37}$$

where T_s is the desired sampling interval. The gain of the resulting discrete-time system $H(z)$ must be corrected by a proper factor.

For the synthesis of continuous zeros and poles using the Oustaloup method with the intent to obtain a discrete-time approximation the transitional frequency ω_h may be chosen such that

$$\omega_h \leqslant \frac{2}{T_s}. \tag{38}$$

Fractional-order integrator: Implementation considerations

We now address the issue of implementing the fractional-order integrator component. A continuous-time integrator of order λ has to be implemented as

$$G_I(s) = \frac{1}{s^\lambda} = \frac{s^{1-\lambda}}{s}$$

to ensure a nice control effect at lower frequencies.

Fractional-order integrator: Implementation considerations

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$$G_I(s) = \frac{1}{s^\lambda} = \frac{s^{1-\lambda}}{s}$$

to ensure a nice control effect at lower frequencies. Its discrete-time equivalent is given by

$$H_I(z^{-1}) = H^{1-\lambda}(z^{-1}) \cdot H_I(z^{-1}), \tag{39}$$

where $H^{1-\lambda}(z)$ is computed using the method presented above, and

$$H_I(z^{-1}) = \frac{T_s}{(1 - z^{-1})} \tag{40}$$

is a simple discrete-time integrator.

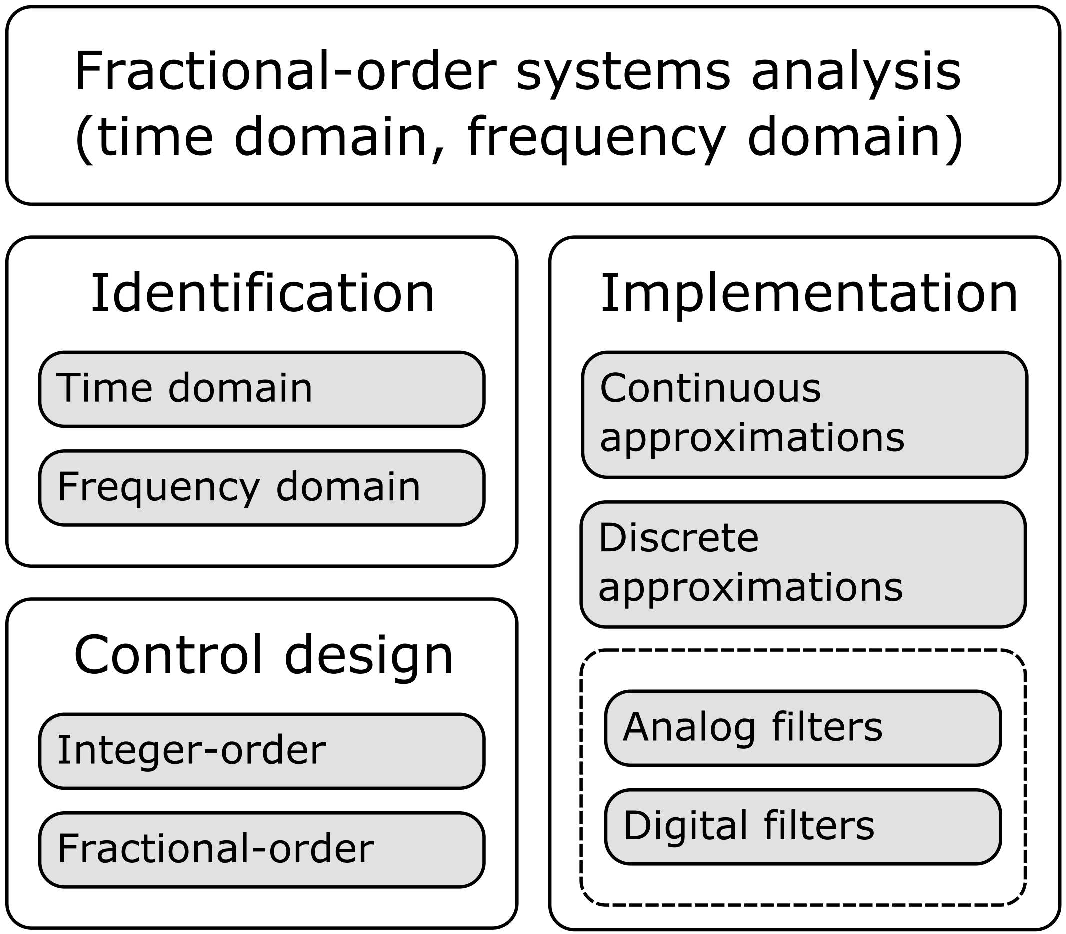
Part IV: CACSD Tools and Applications of Fractional-order Control

FOMCON project: Fractional-order Modeling and Control

FOMCON

- Official website: `http://fomcon.net/`
- Toolbox for MATLAB available;
- An interdisciplinary project supported by the Estonian Doctoral School in ICT and Estonian Science Foundation grant nr. 8738.

FOMCON toolbox: Structure

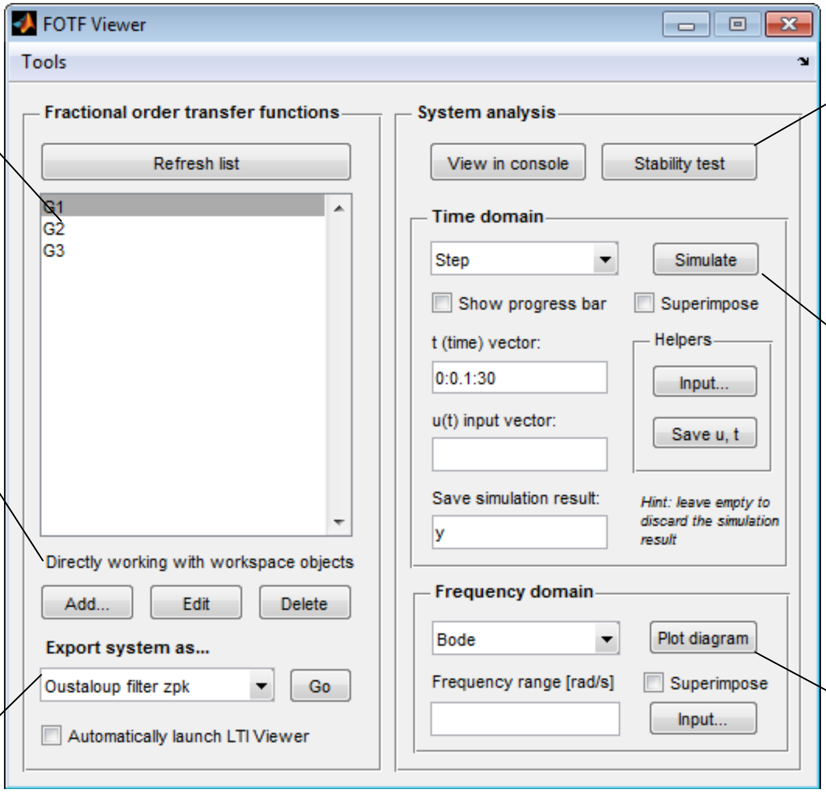


FOMCON toolbox: FOTF Viewer

FOTF systems
in MATLAB workspace

Add, Edit or Delete
FOTF systems

Export FOTF systems
to other formats

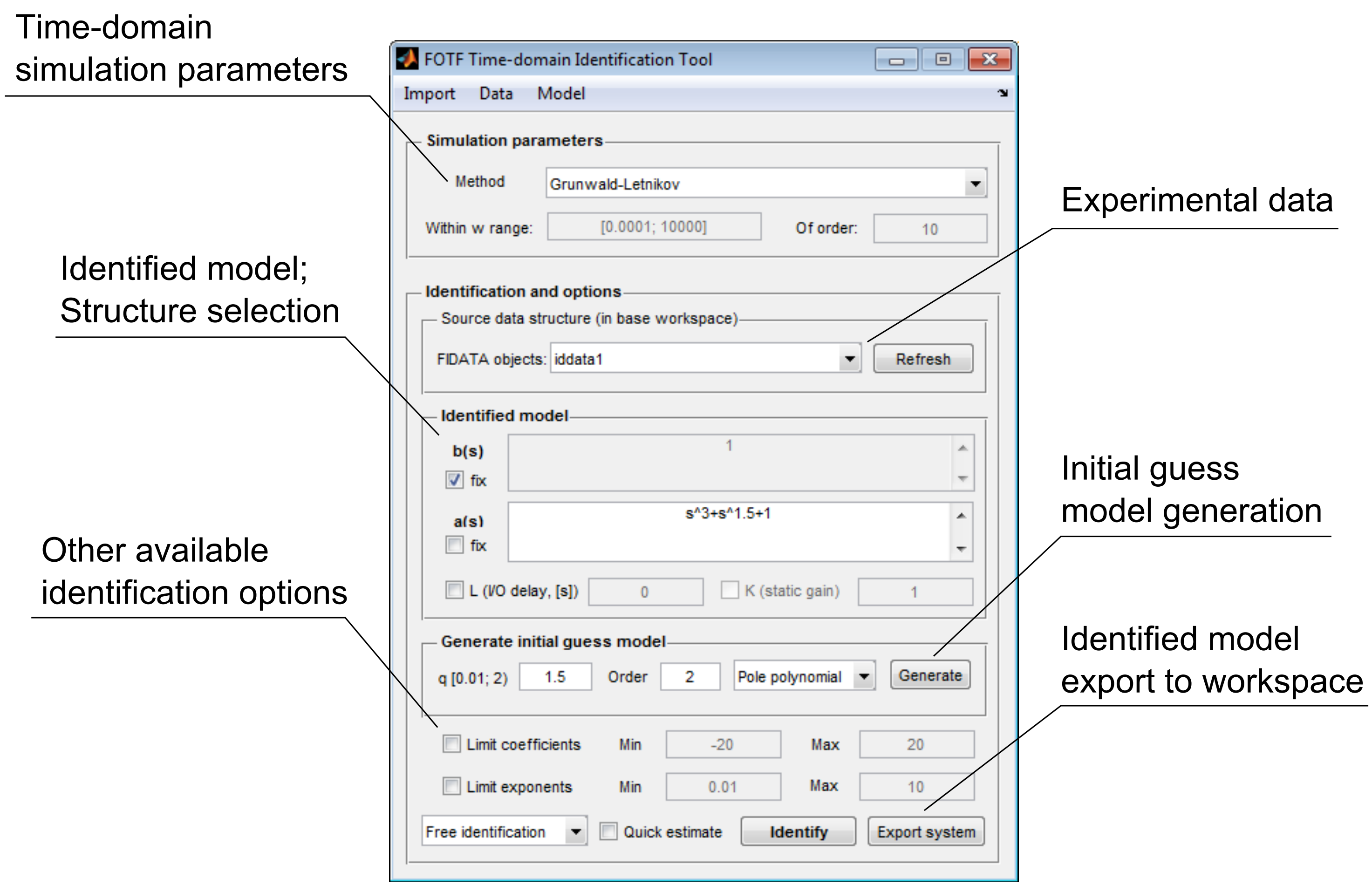


Stability test

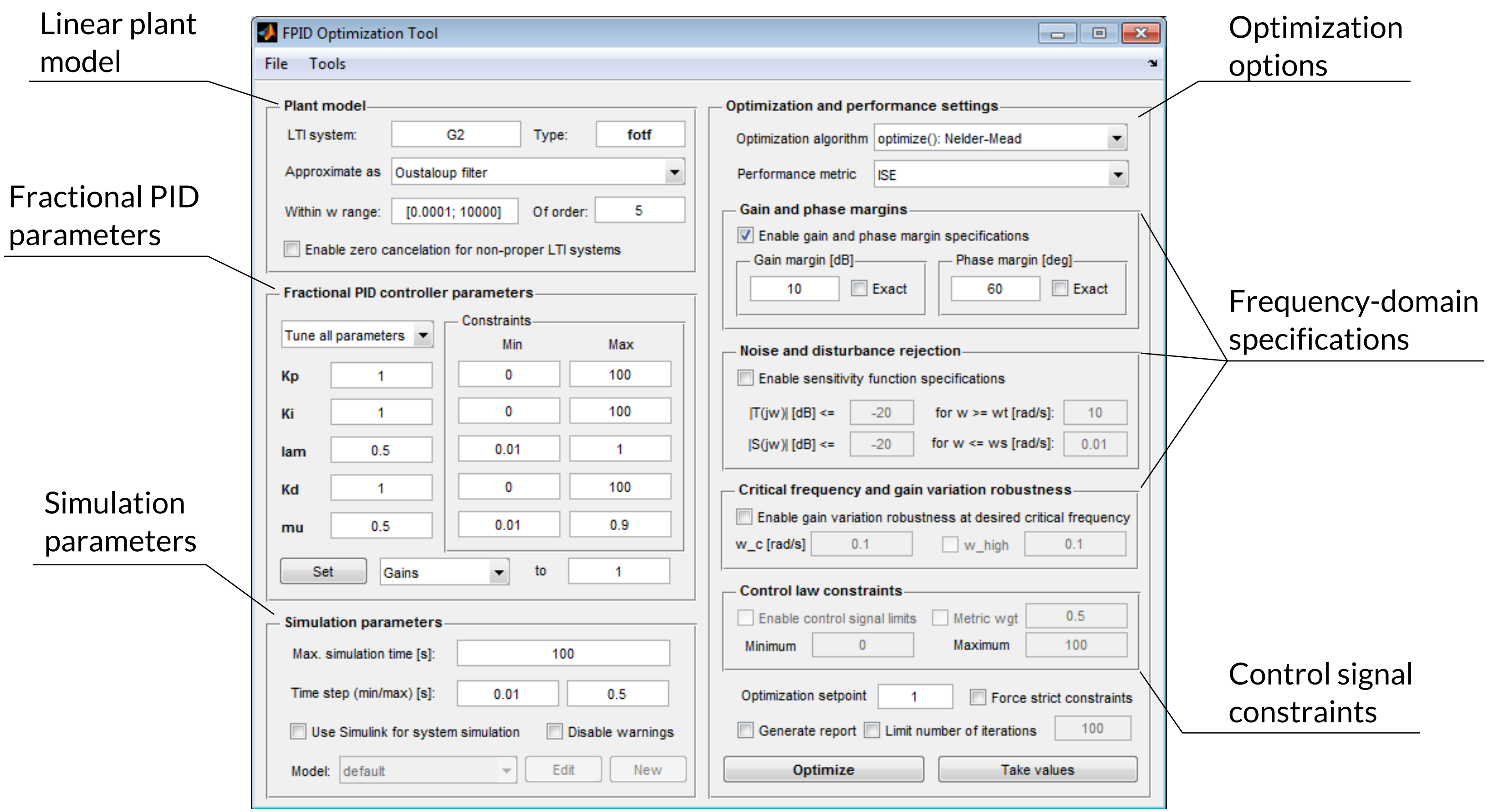
Time-domain
analysis

Frequency-domain
analysis

FOMCON toolbox: Time-domain identification



FOMCON toolbox: Optimization based PI$^\lambda$D$^\mu$ tuning



Example: Fractional-order control of the coupled tanks system

The system is modeled in continuous time in the following way:

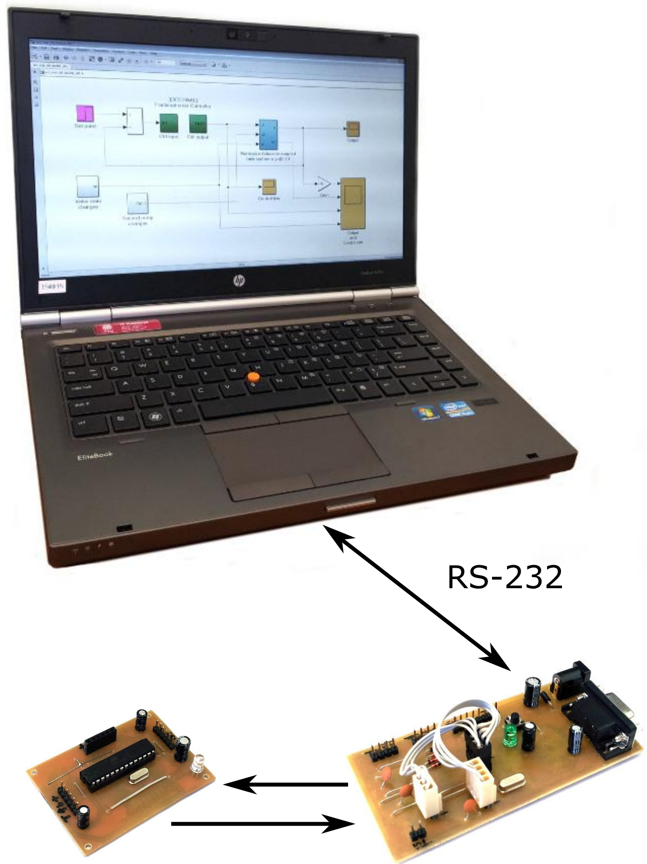
$$\dot{x}_1 = \frac{1}{A}u_1 - d_{12} - w_1 c_1 \sqrt{x_1}, \qquad (41)$$

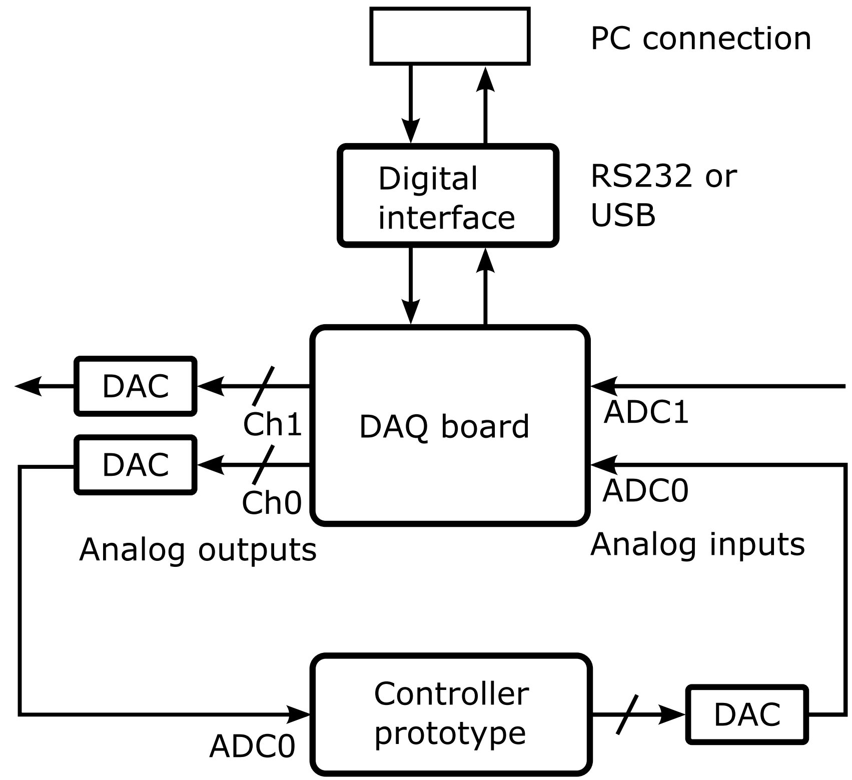
$$\dot{x}_2 = \frac{1}{A}u_2 + d_{12} - w_2 c_2 \sqrt{x_2},$$

where x_1 and x_2 are levels of fluid, A is the cross section of both tanks; c_1, c_2, and c_{12} are flow coefficients, u_1 and u_2 are pump powers; valves are denoted by $w_i : w_i \in \{0, 1\}$ and

$$d_{12} = w_{12} \cdot c_{12} \cdot \mathrm{sign}(x_1 - x_2)\sqrt{|x_1 - x_2|}.$$

Example: Experiments with controller implementation: Hardware platform





Example: Fractional-order control of the coupled tanks system (continued)

Our task is to control the level in the first tank. We identify the real plant from a step experiment with $w_1 = w_{12} = 1, w_2 = 0$ in (41). The resulting fractional-order model is described by a transfer function

$$G_2 = \frac{2.442}{18.0674 s^{0.9455} + 1} \mathrm{e}^{-0.1s}. \qquad (42)$$

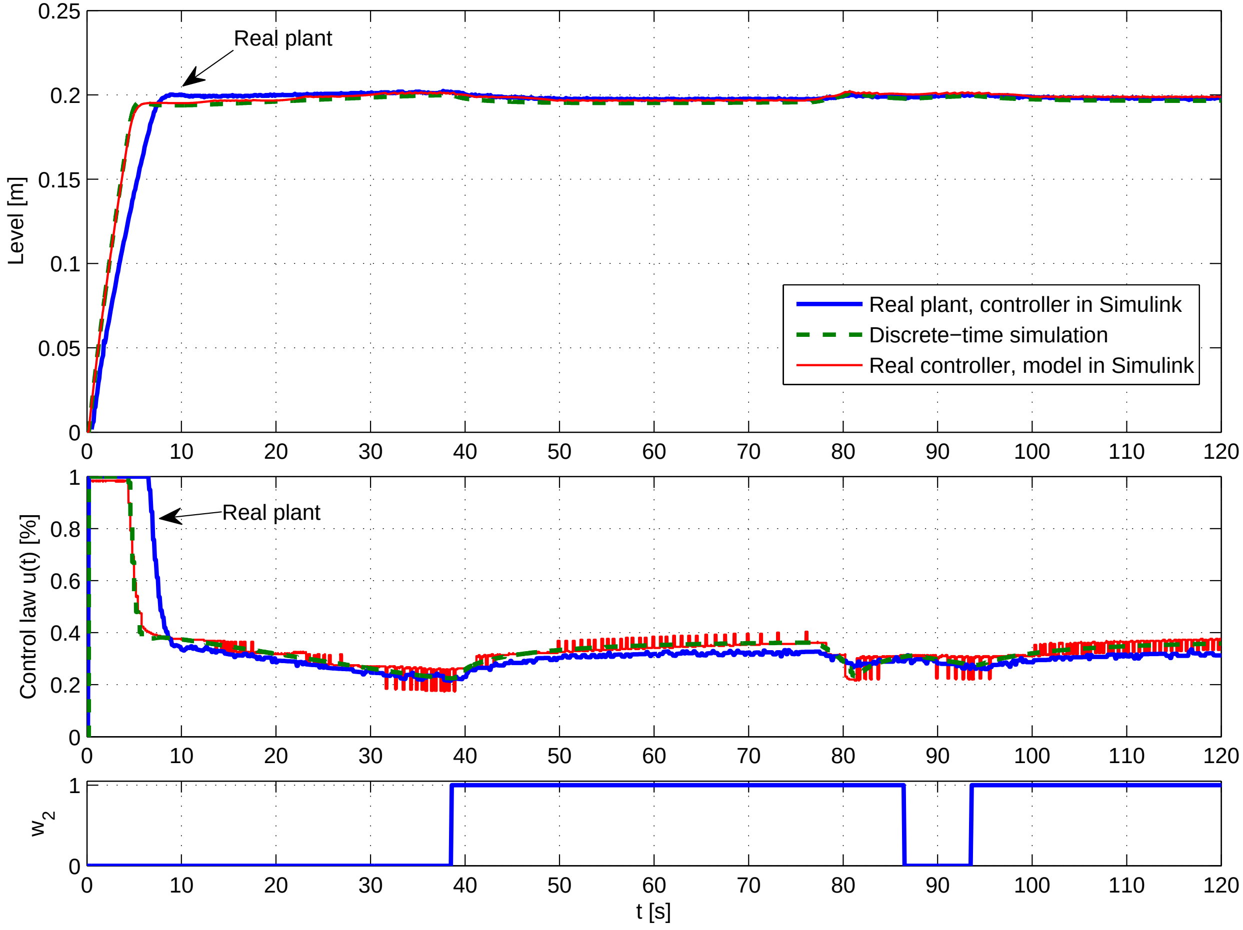
We notice, that this model does not tend to exhibit integer-order dynamics. Due to the value of the delay term the basic tuning formulae for integer-order PID tuning do not provide feasible results. It is possible to select some starting point manually and run optimization several times. However, it is important to choose the correct frequency domain specifications to ensure control system stability.

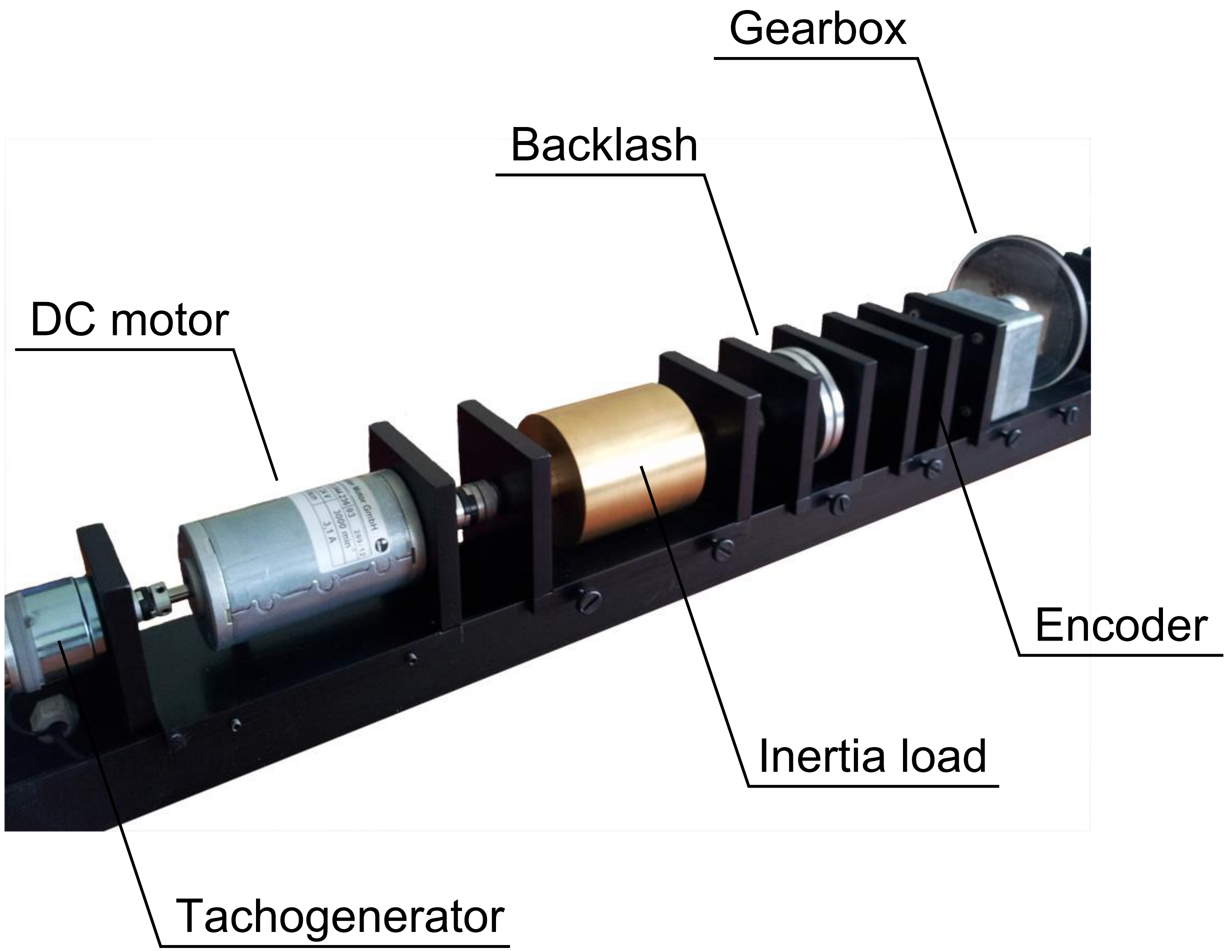
Example: Fractional-order control of the coupled tanks system (continued)

In our case the goal is to minimize the impact of disturbance, so constraints on the sensitivity functions could be imposed. Our choice is such that $\left|T(j\omega)\right| \leq -20\,dB, \forall\omega \geq 10\,\mathrm{rad/s}$ and $\left|S(j\omega)\right| \leq -20\,dB, \forall\omega \leq 0.1\,\mathrm{rad/s}$. The gain and phase margins are set to $G_m = 10\,\mathrm{dB}$ and $\varphi = 60^\circ$, respectively. Additionally, in order to limit the overshoot, the upper bound of control signal saturation was lowered from 100% to 60%. Using the IAE performance metric we finally arrive at the following PI$^\lambda$D$^\mu$ controller parameters by optimizing the response of the nonlinear system in Simulink:

$$K_p = 6.9514, \quad K_i = 0.13522, \quad K_d = -0.99874, \\ \lambda = 0.93187, \quad \mu = 0.29915. \tag{43}$$

Example: Fractional-order control of the coupled tanks system (continued)




Gearbox
Backlash
DC motor
Encoder
Inertia load
Tachogenerator

Example: Fractional-order control of a servo system (continued)

The following transfer function is identified:

$$G(s) = \frac{192.1638}{s(1.001s + 1)}.$$

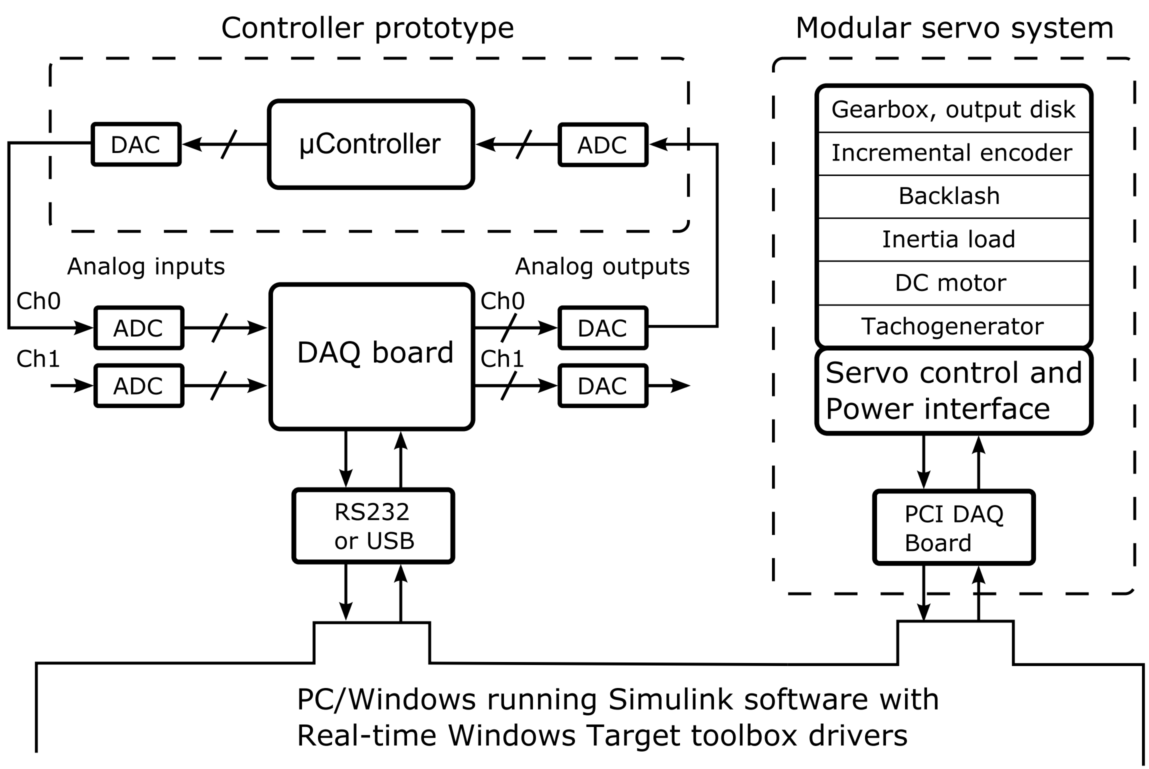
The generic PD controller parameters provided by INTECO are $K_p = 0.1, K_d = 0.01$. We shall use these parameters as the initial ones for the optimization.

The results of optimization are such, that after 100 iterations the gains of the PD controller have been found as $K_p = 0.055979$ and $K_d = 0.025189$.

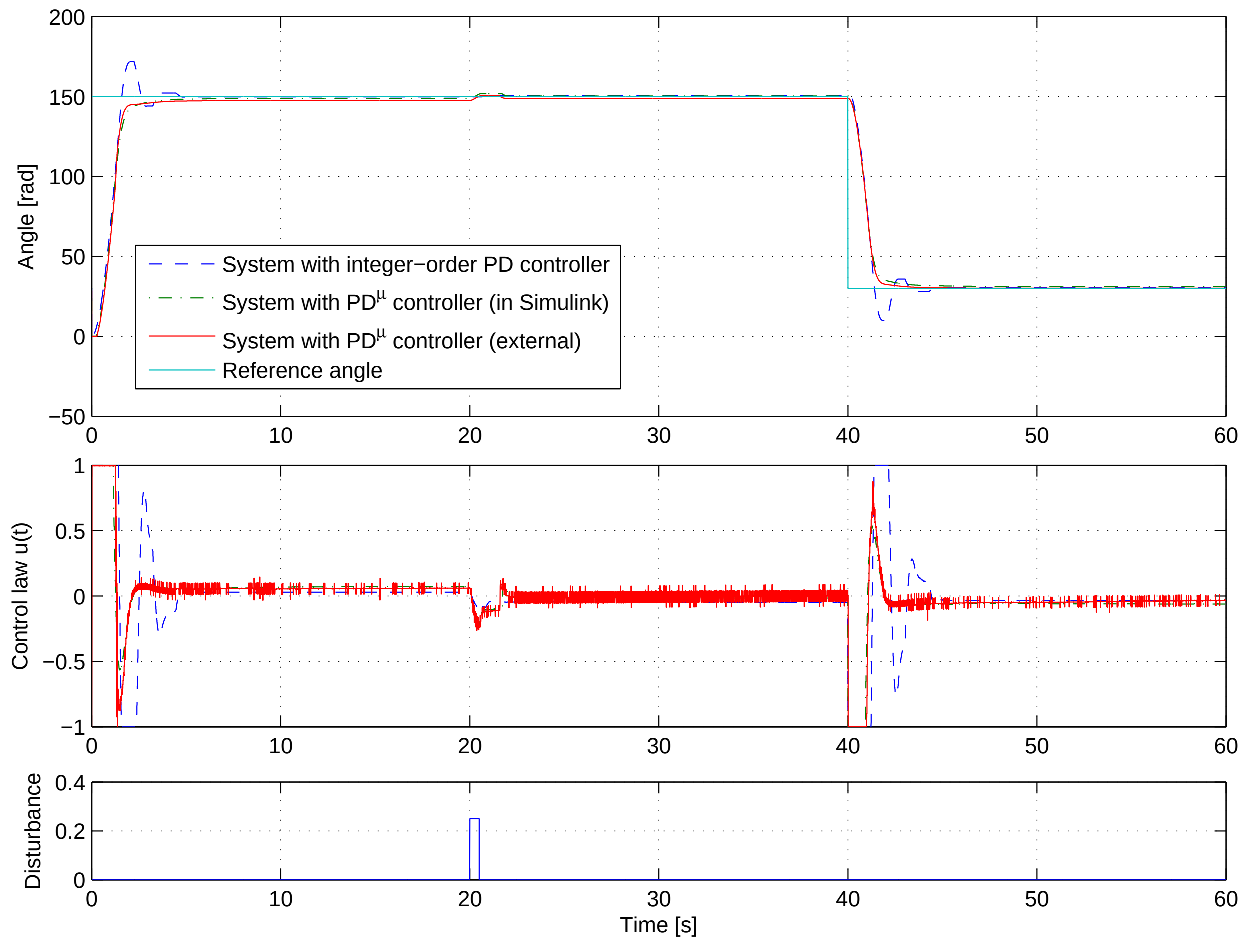
After fixing the gains and manually perturbing the value of μ to 0.5, the optimized PD$^\mu$ controller is obtained with $\mu = 0.88717$. Phase margin of the open loop control system is $\varphi_m = 65.3°$.

Example: Fractional-order control of a servo system (continued)

Experimental setup for evaluating the digital implementation of the fractional-order PID controller:



Example: Fractional-order control of a servo system (continued)



Example: Electrical network approximations

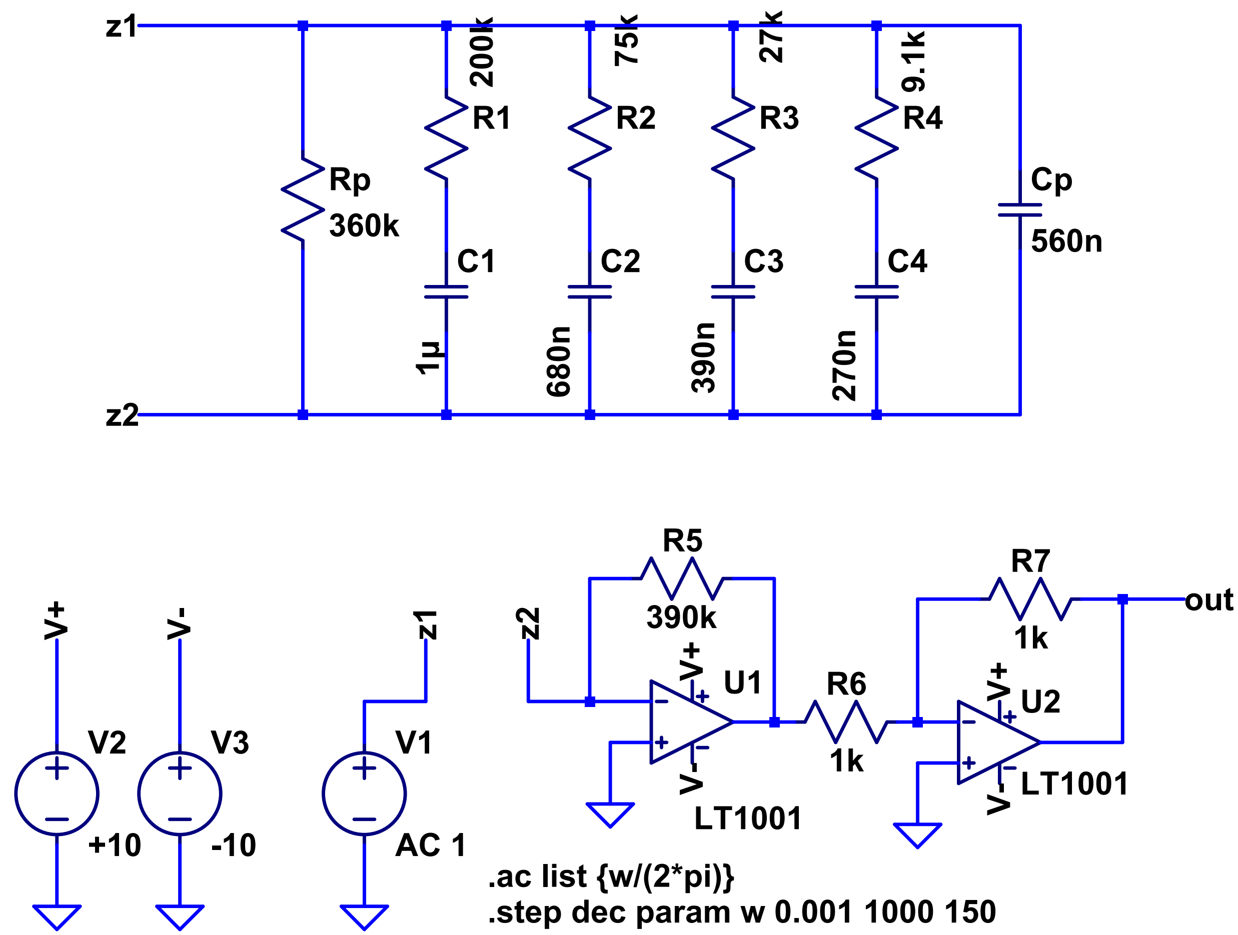
In this example, our goal is to obtain an analog implementation a fractional controller for a model of a position servo

$$G(s) = \frac{1.4}{s(0.7s+1)}\mathrm{e}^{-0.05s}.$$

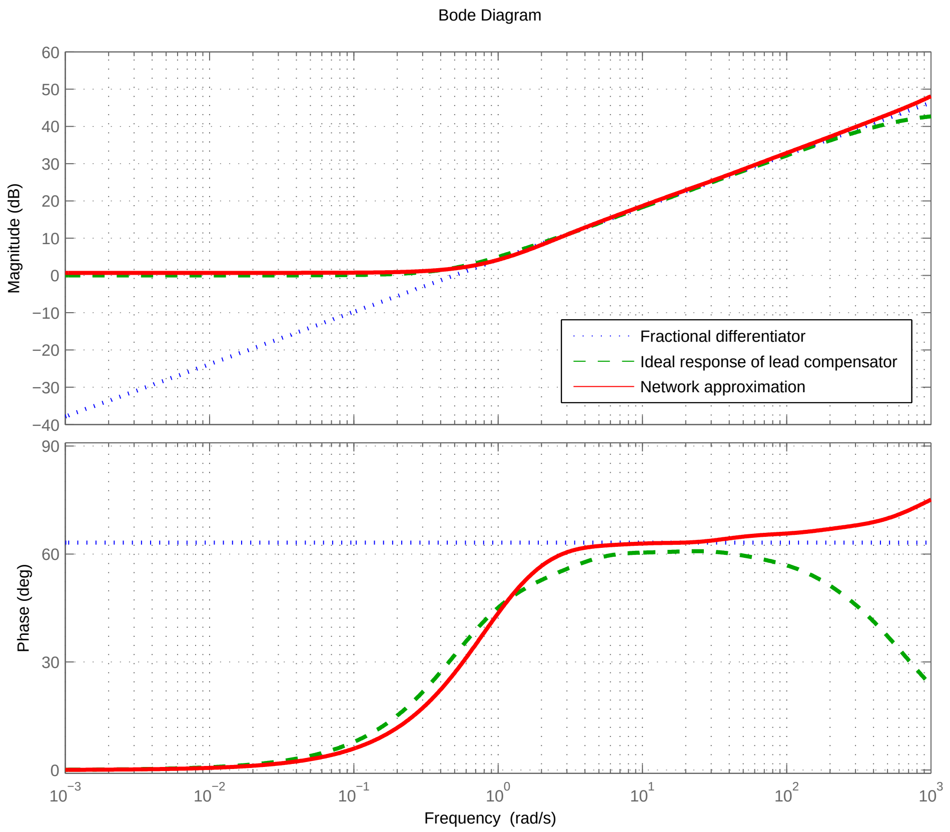
The design specifications are as follows: phase margin $\varphi = 80^\circ$, gain crossover frequency $\omega_{cg} = 2.2$ rad/s. The proposed controller design, based on robustness considerations, is derived from the desired frequency domain characteristics of the plant, in the form of a fractional lead compensator ($\approx$PD$^\mu$ controller):

$$C(s) = \left(\frac{2.0161s+1}{0.0015s+1}\right)^{0.7020}.$$

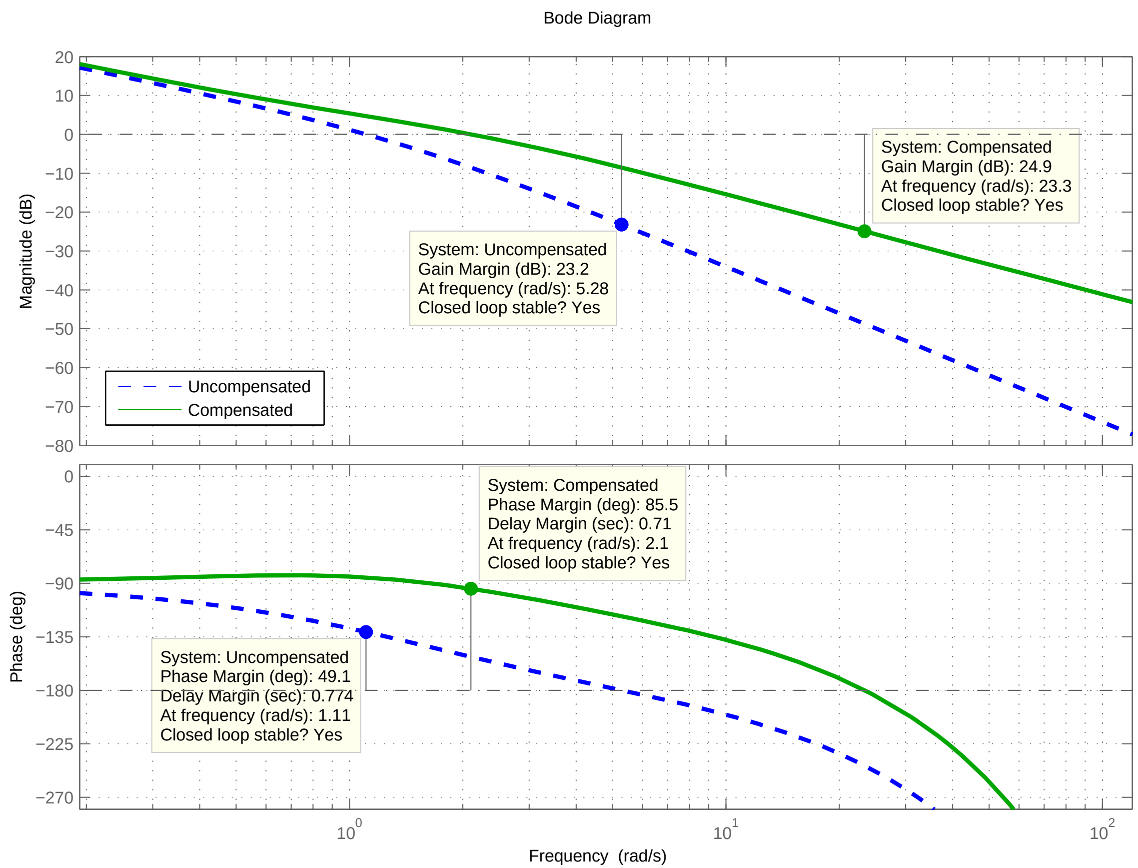
Example: Electrical network approximations (continued)



Example: Electrical network approximations (continued)



Example: Electrical network approximations (continued)



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