Various notions of linearization of nonlinear control systems: concepts, methods, applications

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Summary

1. Introduction
2. State-space equivalence and linearization
3. Feedback equivalence and linearization
4. Orbital feedback equivalence and linearization

Class of control systems

- finite-dimensional
- smooth
- time-continous

We will consider

$$\Xi \ : \ \dot{x} = F(x, u)$$

- $x \in X$, state space, an open subset of $\mathbb{R}^n$
- $u \in U$, set of control values, a subset of $\mathbb{R}^m$
- F is smooth (C^k or C^∞) with respect to (x, u)

Very often: control-affine systems

$$\Sigma \ : \ \dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x), \quad x \in X \subset \mathbb{R}^n, u \in \mathbb{R}^m$$

- f and $g_1, \ldots, g_m$ are smooth vector fields on X
- state-dependent nonlinearities
- common in applications

Linearization problem

When is Ξ or Σ equivalent (transformable) to a linear control system?

- define equivalence (or the class of transformations)
- find conditions for linearization
- construct linearizing transformations

Summary

The system

$$\Xi \;:\; \dot{x} = F(x,u),\; x \in X,\; u \in U \;\text{ and}$$

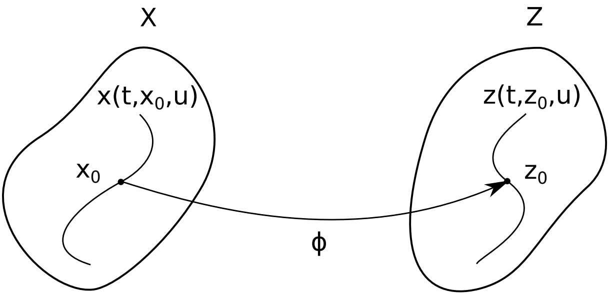
$$\tilde{\Xi} \;:\; \dot{z} = \tilde{F}(z,u),\; z \in Z,\; u \in U \;\text{ the same control}$$

are state-space equivalent, shortly S-equivalent, if there exists a diffeomorphism $z = \Phi(x)$ such that

$$\frac{\partial \Phi}{\partial x} \cdot F(x,u) = \tilde{F}(\Phi(x),u).$$

- the Jacobian matrix of Φ (the derivative of Φ) maps the dynamics F of Ξ into $\tilde{F}$ of $\tilde{\Xi}$
- A diffeomorphism is a map Φ such that
  - Φ is bijective
  - Φ and Φ^{-1} are C^k (C^∞)
- A (local) diffeomorphism defines a (local) nonlinear change of coordinates $z = \Phi(x)$

S-equivalence preserves trajectories



The image under Φ of a trajectory of Ξ is a trajectory of $\tilde{\Xi}$ (corresponding to the same control.

S-linearization

Problem 1 When is Σ S-equivalent to a linear system, i.e., when does there exist $z = \Phi(x)$ transforming Σ into a linear system of the form

$$\dot{z} = Az + \sum_{i=1}^{m} u_i b_i, \quad x \in \mathbb{R}^n$$

that is, for $1 \leq i \leq m$,

$$\frac{\partial \Phi}{\partial x}(x) \cdot f = Az \quad \text{and} \quad \frac{\partial \Phi}{\partial x}(x) \cdot g_i(x) = b_i$$

- We want the same diffeomorphism Φ to transform f into Az (a linear vector field) and g_i into b_i, for $1 \leq i \leq m$ (constant vector fields)

Why is S-linearization interesting?

- If we want to solve a control problem for Σ and Σ is S-equivalent to a linear system Λ, then
- transform Σ into Λ
- solve the problem for the linear system Λ
- transform the solution (via the inverse Φ^{-1} of Φ)
- examples: point-to-point controllability, trajectory tracking, stabilizability (we will say more when talking about feedback linearization)

A little bit of geometry: Lie bracket

Given two vector fields f and g on X, we define their Lie bracket as

$$[f,g](x) = \frac{\partial g}{\partial x}(x)f(x) - \frac{\partial f}{\partial x}(x)g(x)$$

It is a new vector field on X. How to interpret it?

Consider the control system

$$\dot{x} = u_1 f(x) + u_2 g(x),$$

and apply the control strategy

$$\begin{aligned} u_1 &= 1 \\ u_2 &= 0 \end{aligned} \qquad s \in [0, t]$$

$$\begin{aligned} u_1 &= 0 \\ u_2 &= 1 \end{aligned} \qquad s \in [t, 2t]$$

$$\begin{aligned} u_1 &= -1 \\ u_2 &= 0 \end{aligned} \qquad s \in [2t, 3t]$$

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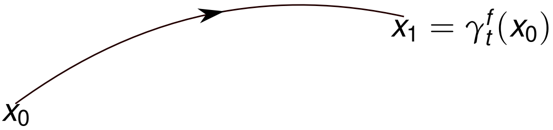
$$\begin{array}{rcl} u_1 & = & 0 \\ u_2 & = & -1 \end{array} \quad s \in [3t, 4t]$$

Denote by $\gamma_t^f(x_0) = x(t, x_0)$ the solution of the differential equation $\dot{x} = f(x)$ and by $\gamma_t^g(x_0) = x(t, x_0)$ the solution of the differential equation $\dot{x} = g(x)$, passing through x_0.



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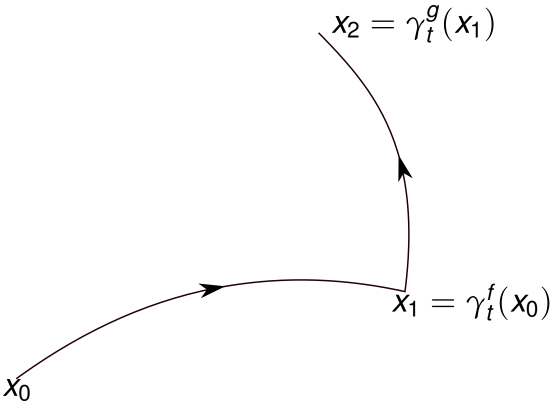

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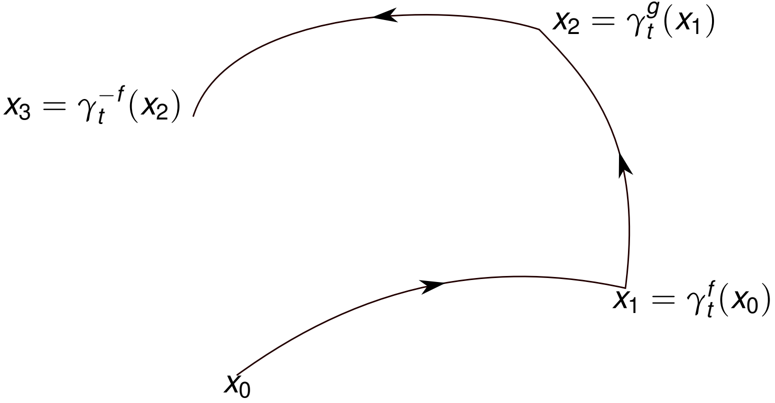

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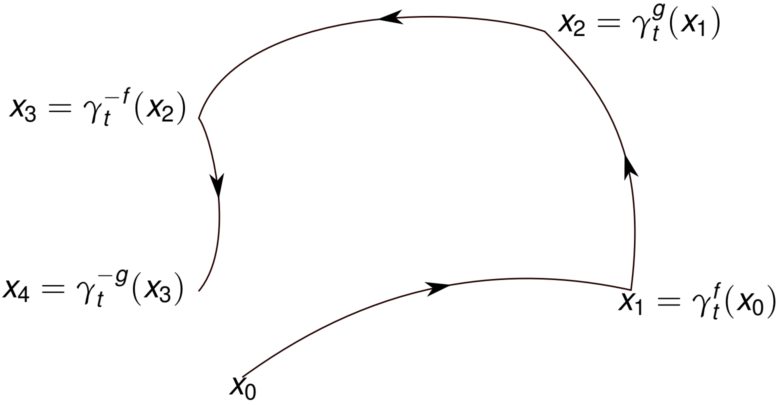

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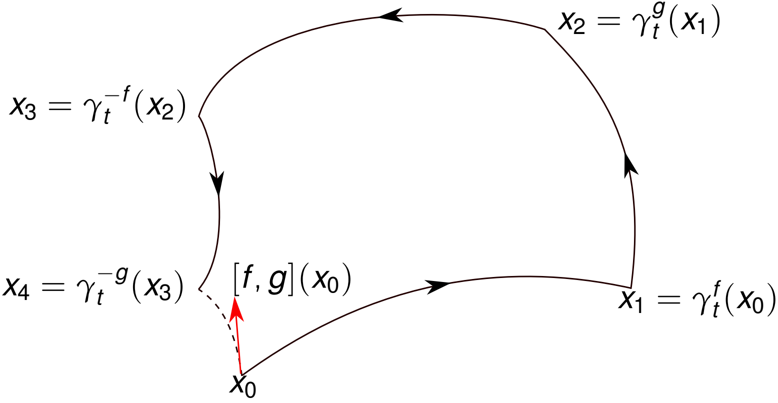


Define

$$\begin{aligned} ad_f^0 g &= g \\ ad_f g &= [f, g] \\ \text{and, inductively } ad_f^k g &= [f, ad_f^{k-1} g] = [f, \ldots, [f, g],] \end{aligned}$$

For the single-input system

$$\dot{x} = f(x) + ug(x)$$

the Lie bracket $ad_f g = [f, g] = [f, f + g]$ measures to what extent the trajectories of f (corresponding to $u \equiv 0$) do not commute with those of $f + g$ (corresponding to $u \equiv 1$).

Theorem

Σ is, locally around x_0, S-equivalent to a controllable linear system Λ if and only if

(SL1) $\text{span}\left\{ad_f^q g_i(x_0) : \ 1 \le i \le m, \ 0 \le q \le n-1\right\} = \mathbb{R}^n$

(SL2) $[ad_f^q g_i, ad_f^r g_j] = 0, \ \text{for}\ 1 \le i,j \le m, \ 0 \le q,r \le n$

- Interpretation
  - (SL1) guarantees controllability of Λ
  - (SL2) implies that all iterative Lie brackets containing at lest two g_i's vanish
- Verification
  - (SL1) and (SL2) are verifiable in terms of f and g_i's using differentiation and algebraic operations only (no need to solve PDE's)

Constructing linearizing coordinates

Assume, for simplicity, the scalar-input case $m = 1$. In order to find the linearizing diffeomorphism $z = \Phi(x)$ solve the system of n 1st order PDE's:

$$(S) \quad \frac{\partial \Phi}{\partial x} A(x) = Id,$$

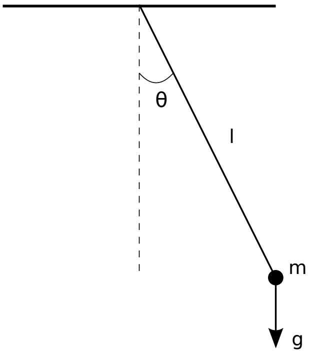
where $A(x) = (A_1(x), \ldots, A_n(x))$ and $A_q(x) = ad_f^{q-1} g(x)$, for $1 \leq q \leq n$.
(SL2) form the integrability conditions for (S) and assure the existence of solutions.

- Do not confuse S-linearization with linear approximation
- Assume $F(x_0, u_0) = 0$. The linear approximation of $\dot{x} = F(x, u)$ is

$$\begin{aligned}\dot{z} &= Az + Bv + \textit{higher order terms} \\ \dot{z} &= Az + Bv,\end{aligned}$$

where $A = \frac{\partial F}{\partial x}(x_0, u_0)$ and $B = \frac{\partial F}{\partial u}(x_0, u_0)$
- So we neglect (erase) higher order terms
- In S-linearization higher order terms are compensated via the diffeomorphism Φ (no terms are neglected)

Consider the pendulum



The states are $(x_1, x_2) = (\theta, \dot{\theta})$ and the control is the torque u

The equations are

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l}\sin x_1 + \frac{1}{ml^2}u. \end{aligned}$$

We have

$$f = \begin{pmatrix} x_2 \\ -\frac{g}{l}\sin x_1 \end{pmatrix}, \quad g = \begin{pmatrix} 0 \\ \frac{1}{ml^2} \end{pmatrix}, \quad ad_f g = -\begin{pmatrix} \frac{1}{ml^2} \\ 0 \end{pmatrix}$$

yielding

$$[g, ad_f g] = 0 \text{ but } [ad_f g, ad_f^2 g] \neq 0$$

which implies that the pendulum is not S-linearizable

- But put $u = ml^2(\frac{g}{l}\sin x_1 + v)$
- we get the linear controllable system (in the Brunovsky form)

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= v. \end{aligned}$$

- therefore there systems that become linear after applying a (nonlinear) transformation in the control space

Summary

The systems

$$\Xi \ : \ \dot{x} = F(x,u), \ x \in X, \ u \in U \ \text{ and}$$

$$\tilde{\Xi} \ : \ \dot{z} = \tilde{F}(z,v), \ z \in Z, \ u \in V \ \text{ not the same control}$$

are feedback equivalent, shortly F-equivalent, if there exists

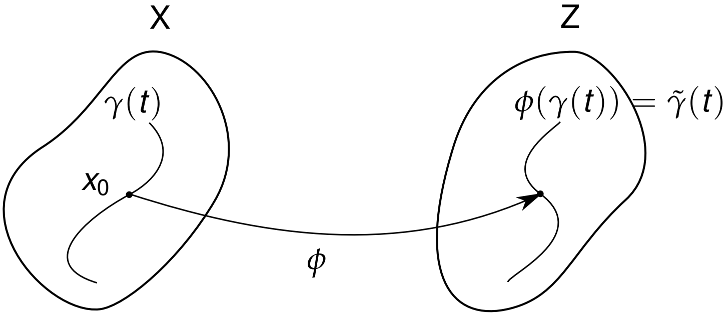
- a diffeomorphism $z = \Phi(x)$ and
- a control transformation $v = \Psi(x,u)$, invertible with respect to u

such that

$$\frac{\partial \Phi}{\partial x} \cdot F(x,u) = \tilde{F}(\Phi(x), \Psi(x,u)).$$

Why is F-equivalence interesting?

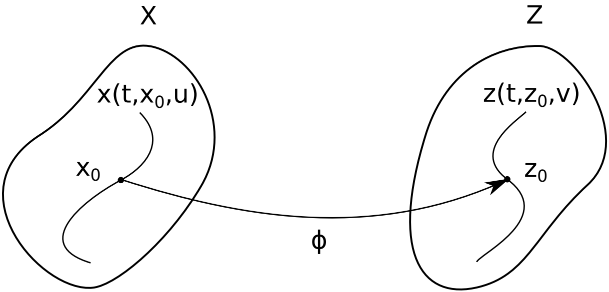
Does F-equivalence preserve trajectories?



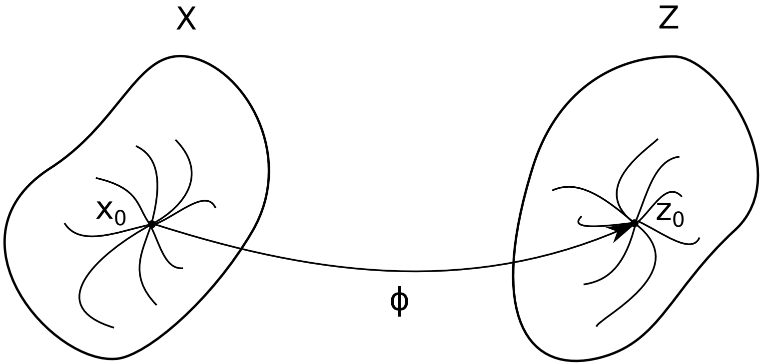
Is the image of a trajectory, via the diffeomorphism $z = \Phi(x)$, a trajectory?

Yes, the image of a trajectory of Ξ, for a control $u(t)$, is a trajectory of $\tilde{\Xi}$ corresponding to

$$v(t) = \Psi(x(t), u(t))$$

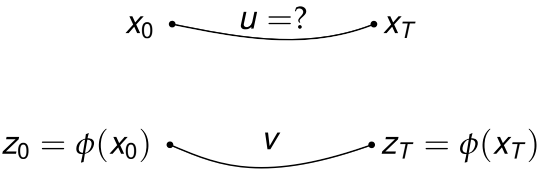


Therefore, F-equivalence preserves the set of all trajectories (the totality of trajectories)

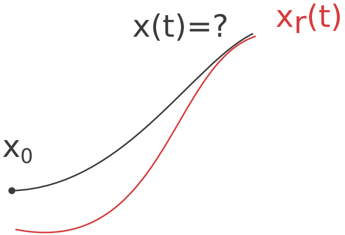


F-equivalence is thus interesting for all problems that depend on the set of all trajectories (and not on a particular parametrization with respect to control. Examples of such problems are:

Point-to-point controllability



Trajectory tracking



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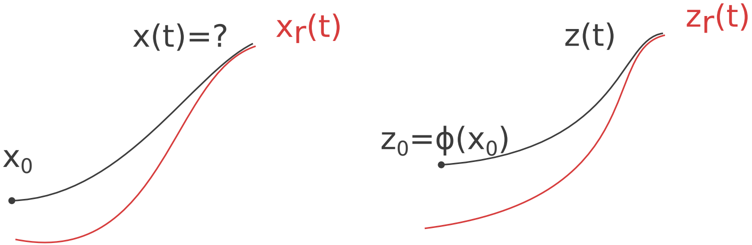
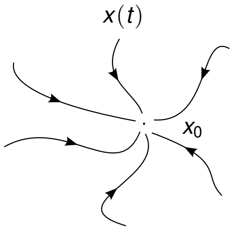


Stabilization



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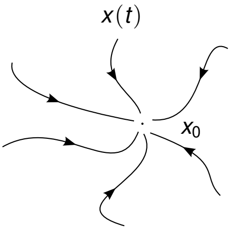

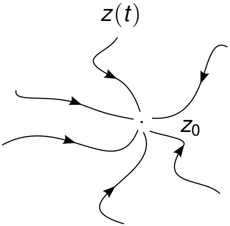

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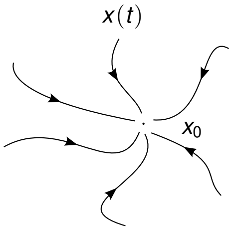
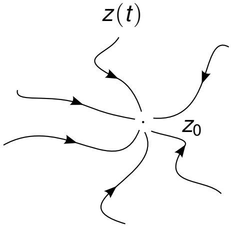
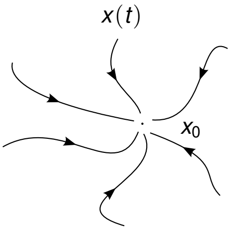


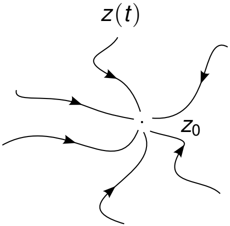


$v = \sigma(z)$ stabilizes (asymptotically)

$\dot{z} = (f + g\alpha) + g\beta v$

Stabilization





$u = \alpha + \beta\sigma$ stabilizes (asymptotically) $\Leftarrow$ $v = \sigma(z)$ stabilizes (asymptotically)

$\dot{x} = f + gu$ $\qquad$ $\dot{z} = (f + g\alpha) + g\beta v$

Problem 2 When is Ξ F-equivalent to a linear system, i.e., when do there exist $z = \Phi(x)$ and $\Psi(x, u)$ transforming Ξ into a linear system of the form

$$\dot{z} = Az + \sum_{i=1}^{m} u_i b_i, \quad x \in \mathbb{R}^n?$$

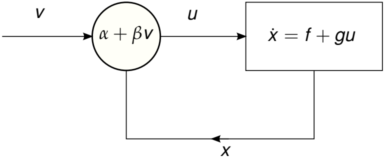
for control affine systems

$$\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x), \quad x \in X$$

we apply $z = \Phi(x)$ and control-affine feedback transformation

$$u = \alpha(x) + \beta(x)v,$$

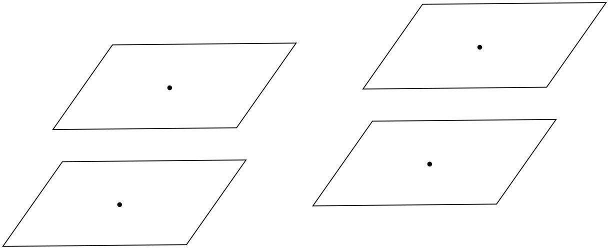
where the matrix β is invertible.



More geometry: a distribution is a map assigning to any $x \in X$

$$x \longmapsto \mathcal{D}(x),$$

a linear subspace of the tangent space (the space of all tangent vectors at x or, equivalently, the space of all velocities at x)



- very often $\mathcal{D} = \text{span}\{f_1, \ldots, f_k\}$ is spanned by vector fields
- $\mathcal{D}$ is involutive if $[f_i, f_j] \in \mathcal{D}$, for any $1 \le i, j \le k$
- Put $\mathcal{D}^j = \text{span}\{ad_f^q g_i;\ 1 \le i \le m,\ 0 \le q \le j-1\}$

Theorem

Σ is, locally around x_0, F-equivalent to a controllable linear system Λ if and only if

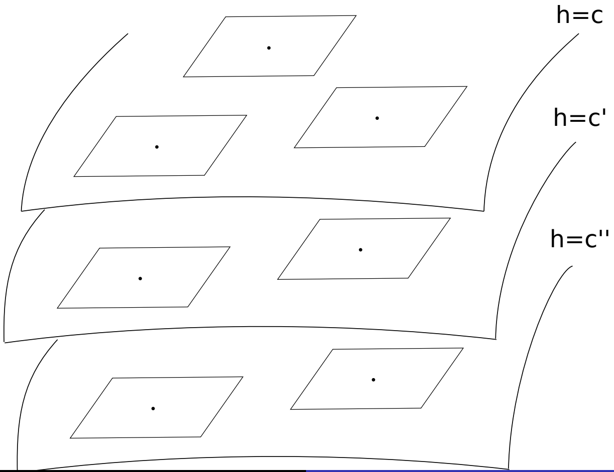
(FL1) dim $\mathcal{D}^j(x)$*=const.*

(FL2) dim $\mathcal{D}^n(x) = n$

(FL3) $\mathcal{D}^j$ *are involutive, for* $0 \le j \le n$

- (FL2) guarantees controllability of Λ
- (FL1)-(FL3) are verifiable in terms of f and g_i's using differentiation and algebraic operations only (no need to solve PDE's)

Assume, for simplicity $m = 1$. Involutivity of $\mathcal{D}^{n-1}$ (of dimension $n-1$ at any x) is equivalent to the existence of a family of hypersurfaces $H_c = \{x \in X : \ h(x) = c\}$ tangent to $\mathcal{D}^{n-1}$



Constructing linearizing transformations

- The normal vector to the hypersurface H_c has to be annihilated by $g, \dots, ad_f^{n-2} g$ spanning $\mathcal{D}^{n-1}$. So solve

$$(S) \quad \frac{\partial h}{\partial x} A(x) = 0, \text{where } A(x) = (g(x), \dots, ad_f^{n-2} g(x))$$

- any solution h of (S) gives linearizing coordinates

$$z_i = L_f^{i-1} h, \text{ for } 1 \leq i \leq n$$

- and linearizing feedback

$$v = L_f^n h + u L_g L_f^{n-1} h$$

Summary

For the system

$$\Xi \, : \, \frac{dx}{dt} = \dot{x} = F(x,u), \; x \in X, \; u \in U$$

define a new time scale τ such that

$$\frac{dt}{dx} = \gamma(x(t)),$$

where γ is a nonvanishing function on X. With respect to the new time scale τ

$$\frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = \gamma(x)F(x,u).$$

The systems

$$\Xi \;:\; \dot{x} = F(x,u),\;\; x \in X,\;\; u \in U \;\; \text{and}$$

$$\tilde{\Xi} \;:\; \dot{z} = \tilde{F}(z,v),\;\; z \in Z,\;\; u \in V \;\; \text{not the same control}$$

are orbitally feedback equivalent, shortly OF-equivalent, if there exists

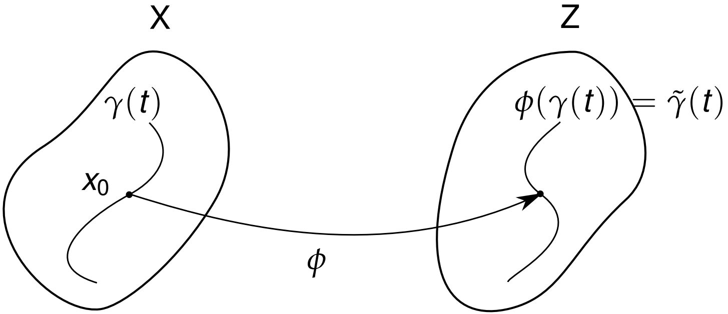
- a diffeomorphism $z = \Phi(x)$ and
- a control transformation $v = \Psi(x,u)$, invertible with respect to u
- a nonvanishing function γ on X

such that

$$\frac{\partial \Phi}{\partial x} \cdot \gamma(x) F(x,u) = \tilde{F}(\Phi(x), \Psi(x,u)).$$

Why is OF-equivalence interesting?

Does OF-equivalence preserve trajectories?

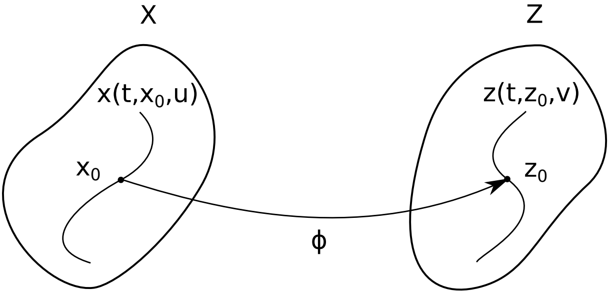


Is the image of a trajectory, via the diffeomorphism $z = \Phi(x)$, a trajectory?

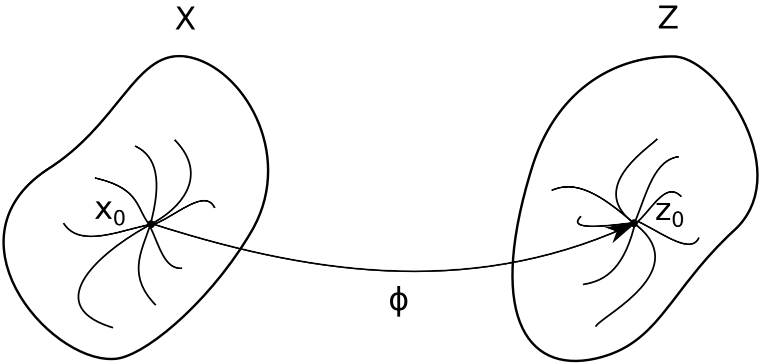
Yes, the image of a trajectory of Ξ, for a control $u(t)$, is a trajectory of $\tilde{\Xi}$ corresponding to

$$v(t) = \Psi(x(t), u(t))$$

and parameterized by the new time $\tau = \gamma(x(t))$



Therefore, OF-equivalence preserves the set of all trajectories (the totality of trajectories) as unparameterized curves



OF-equivalence is thus interesting for all problems that depend on the set of all trajectories and not on a particular parametrization with respect to control and time.

Define the distributions

$$\begin{aligned} \mathcal{G} &= \operatorname{span}\{g_1, \dots, g_m\}, \\ \mathcal{G}_f^j &= \operatorname{span}\{f, g_i, \operatorname{ad}_f g_i, \dots, \operatorname{ad}_f^{j-1} g_i, \quad 1 \le i \le m\}, \text{ for } 1 \le j \le n+1. \end{aligned}$$

the differential forms

$$\begin{aligned} \omega^j(h) &= 0, \quad \text{for any } h \in \mathcal{G}_f^n, \\ \omega^j(\operatorname{ad}_f^n g_i) &= \delta_i^j \end{aligned}$$

and the functions:

$$\mathrm{T}_{i,j}^{k,l} = \omega^k([\operatorname{ad}_f^{n-1} g_i, \operatorname{ad}_f^l g_j])$$

From control systems to distributions

To the control system

$$\Sigma \ : \ \dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x),$$

we attach the distributions

$$\mathcal{D}_\Sigma = \mathrm{span}\{f, g_1, \ldots, g_m\} = \mathrm{span}\{f\} + \mathcal{G}_\Sigma.$$

Theorem

The following conditions are equivalent:

- Σ *is, locally around* x_0*, OF-equivalent to a controllable linear system* Λ
- Σ *satisfies*

(OFL1) $\dim \mathcal{G}_f^{n+1}(x) = (n+1)m + 1$;
(OFL2) $[\mathcal{G}_f^j, \mathcal{G}_f^j] \subset \mathcal{G}_f^{j+1}$, for $1 \le j \le n$;
(OFL3) $[\mathcal{G}, \mathcal{G}_f^2] \subset \mathcal{G}_f^2$;
(OFL4) *The functions* $T_{i,j}^{k,l}$ *equal zero or one.*

- Σ *satisfies*

(OFL1)' $\mathcal{C}(\mathcal{D}_\Sigma^{(1)}) = \mathcal{G}_\Sigma$, *where* $\mathcal{C}(\mathcal{D}_\Sigma^{(1)})$ *is the characteristic distribution of* $\mathcal{D}_\Sigma^{(1)} = [\mathcal{D}_\Sigma, \mathcal{D}_\Sigma]$
(OFL2)' $\mathcal{D}_\Sigma$ *is locally equivalent to the chained form.*

- (OFL1)-(OFL4) are generalizations of involutivity conditions for feedback linearization
- (OFL1)-(OFL4) are verifiable in terms of f and g_i's using differentiation and algebraic operations only (no need to solve PDE's)

The *n*-trailer (considered as a control-affine system, i.e, taking backward-forward velocity constant) is orbitally feedback linearizable !!! (ShunJie Li and WR)

n-trailer system

The n-trailer is a flat system: (x_0, y_0) is a flat output but there are many others.

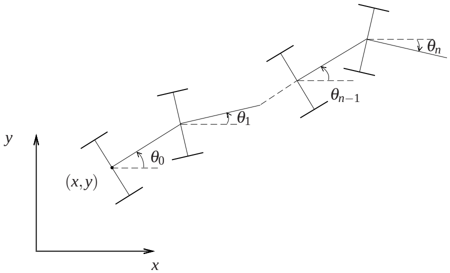


Figure: The n-trailer system

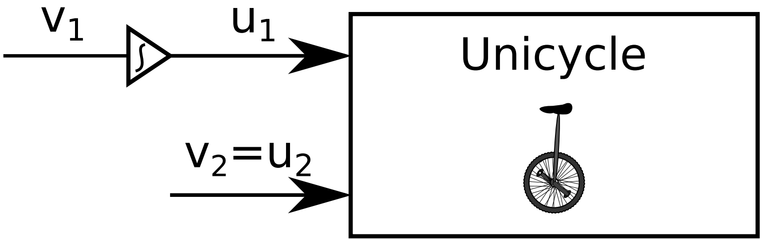
Consider the unicycle

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\theta} \end{pmatrix} = u_1 \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

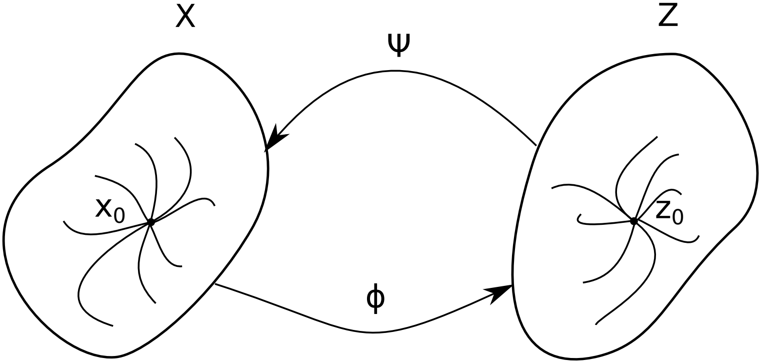
$$[g_1, g_2] = Dg_2 \cdot g_1 - Dg_1 \cdot g_2 = \begin{pmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{pmatrix} \notin \mathcal{D}^1.$$

$\mathcal{D}^1$ is not involutive so the unicycle is not F-linearizable but it become so after applying the dynamic feedback

$$\begin{aligned} \dot{y} &= v_1 \\ u_1 &= y \\ u_2 &= v_2 \end{aligned}$$


v1
u1
Unicycle
v2=u2

How to formalize? Two systems Ξ and $\tilde{\Xi}$ are dynamically equivalent, shortly D-equivalent, if there exits maps Φ and Ψ mapping trajectories onto trajectories and mutually inverse on trajectories



- Ξ is flat if it is D-equivalent to a linear controllable system.
- Equivalently, $\Xi : \dot{x} = F(x, u)$ is flat if there exist m functions $h_i(x, u, \ldots, u^{(p)})$, where m is the number of controls, such that

$$x = \gamma(h, \ldots, h^{(s)})$$
$$u = \delta(h, \ldots, h^{(s)}),$$

where $h = (h_1, \ldots, h_m)$ are called flat outputs.
- Equivalently $\Xi : \dot{x} = F(x, u)$ is flat if there exists a dynamic precompensator, invertible and endogenous,

$$\Theta : \begin{cases} \dot{y} = G(x, y, v), \ y \in Y \subset \mathbb{R}^r, \ v \in V \subset \mathbb{R}^m \\ u = \Psi(x, y, v) \end{cases}$$

such that the precompensated system

$$\Xi \circ \Theta : \begin{cases} \dot{x} = F(x, \Psi(x, y, v)) \\ \dot{y} = G(x, y, v) \end{cases}$$

is S-linearizable

Nonholononic car

The nonholonomic car is a flat system: (x, y) is a flat output but there are many others.

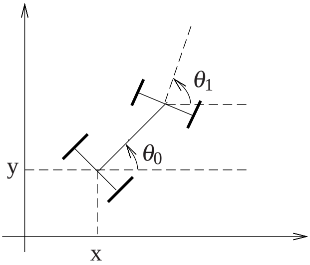


Figure: The nonhonomic car

n-trailer system

The n-trailer is a flat system: (x_0, y_0) is a flat output but there are many others.

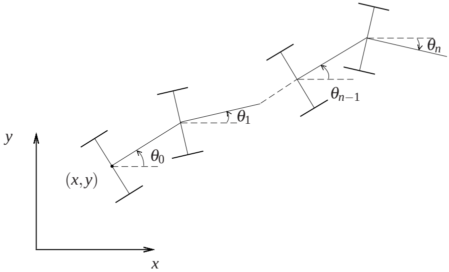
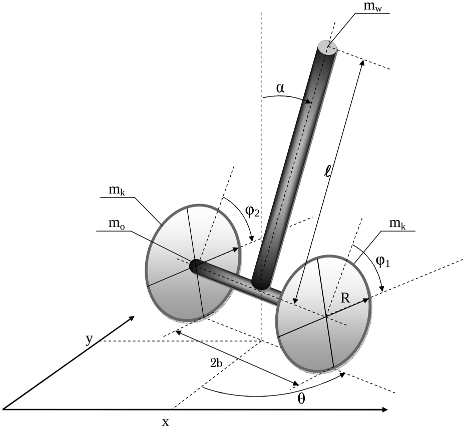


Figure: The n-trailer system


m_w
α
ℓ
m_k
m_o
φ_2
m_k
φ_1
R
y
2b
θ
x

Balancing Robot Modeling

The robot can be described by 6 generalized coordinates $(\mathrm{x}, \mathrm{y}, \theta, \varphi_1, \varphi_2, \alpha) \in \mathbb{R}^2 \times (S^1)^4$, where (x, y) denotes the center of the wheel axle, θ is the orientation angle, φ_1 and φ_2 denote rotation angles of the wheels, and α is the tilt angle. The robot motion is subject to nonholonomic constraints that result in the control-linear system

$$\dot{q} = G(q)\eta, \text{ where } q = (\mathrm{x}, \mathrm{y}, \varphi_1, \varphi_2, \alpha),$$

$$G(q) = \begin{bmatrix} \cos\theta & \cos\theta & \cos\theta \\ \sin\theta & \sin\theta & \sin\theta \\ \frac{2}{R} & 0 & 0 \\ 0 & \frac{2}{R} & 0 \\ 0 & 0 & \frac{1}{R} \end{bmatrix}.$$

The robot orientation can be computed as $\theta = \frac{R}{2b}(\varphi_2 - \varphi_1)$ and $\eta = (\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3$ can be regarded as the control of the system.

Flatness of the Balancing Robot

This system is flat and we can choose the following flat outputs (Kędzierski-Tchoń-Respondek)

$$
\begin{aligned}
y_1 &= \varphi_1 \\
y_2 &= \varphi_2 \\
y_3 &= -R\alpha + \mathrm{x}\cos\theta + \mathrm{y}\sin\theta - \frac{1}{2}R(\varphi_1 + \varphi_2).
\end{aligned}
$$

By a direct calculation, we can express the state variables and controls as suitable functions of the flat outputs and their time derivatives. This can be done out of the singular control value $\dot{\theta} = 0$ (the robot orientation has to vary, i.e., the platform wheels cannot roll with the same angular velocity). This difficulty shows up in control problems of tracking a circular trajectory in the plane.

What do we know about flatness?

- Via flatness we can solve the constructive controllability problem
- Although very useful, flatness is a highly non generic property: a slight perturbation of a flat system yields a non flat one (Tchoń)
- We know that a few classes of control systems are flat: accessible systems with $n-1$ controls, accessible control-linear systems with $n-1$ and $n-2$ controls
- We know to characterize flat control systems of special forms: feedback linearizable systems, control-linear systems with 2 controls (chained form), m-chained form

What don't we know about flatness?

- We do not know to characterize flatness in general.
- We do not know whether the problem is finite or infinite dimensional, that is, we do not know if there is a bound on the number of derivatives of controls
- We do not even know how to check flatness for control-affine systems with 2 controls nor for control-linear systems with 3 controls
- We know that the problem is difficult: E. Cartan (1914) introduced the notion of absolute equivalence of underdetermined differential equations. His absolutely trivial equations are just flat systems. He proved that systems with 2 controls are flat (absolutely trivial) if and only if they are equivalent to the chained form (Goursat normal form). Cartan claimed that the general problem is difficult.
- Non flat systems exist! The first example is due to D. Hilbert (1912)

Conclusions

1. We presented various concepts of linearization: state-space, feedback, dynamic, orbital
2. We provided geometric tools convenient (needed) to study them
3. We presented geometric conditions for solving them (verifiable via differentiation and algebraic operations only), except for dynamic linearizability (flatness)
4. Do not confuse linearization with linear approximation
5. Whenever we can linearize the system, the control problem we are dealing with, get substantially simplified