Lecture 15: Algebraic Methods for Nonlinear Control Systems

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Overview of the talk

- Algebraic framework: basic definitions and constructions
- Polynomial framework
- One nonlinear control problem: Realizability
- Concluding remarks
Two common theories to study nonlinear control systems

- **Differential geometrical approach**: appeared in the 1970s
  A. Isidori, H. Nijmeijer, W. Respondek, A. van der Schaft, etc.

- **Algebraic** methods of differential forms: start from the second half of 1980s
  G. Conte, M. Fliess, Ü. Kotta, C. H. Moog, A. M. Perdon, etc.

**Differential**   **Algebra**

**Calculus and Topology:**
Ordinary differentiation and exterior derivative

**Algebra:**
rings, fields, etc.
Basic definitions: Calculus

Definition (Differentiability)
A real function is said to be differentiable at a point if its derivative exists at that point.

Definition (Derivative)
The derivative of a function $f(x)$ with respect to the variable $x$ is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$ 

Proposition
*If $f(x)$ is differentiable at a point $x_0$, then $f$ is continuous at $x_0$.)*

Example: Function $f(x) = |x|$ is continuous at 0, but not differentiable.
Analytic and meromorphic functions

**Definition**

Analytic function $f(x)$ is an infinitely differentiable function such that the Taylor series at any point $x_0$ in its domain $D$ converges to $f(x)$ for $x$ in a neighborhood of $x_0$ point-wise (and uniformly).

**Examples:** polynomial functions $f(x) = x^2 - 3x + 1$, exponential function $f(x) = e^x$, trigonometric functions $f_1(x) = \cos x$, $f_2(x) = \tanh(3x)$.

**Definition**

If $I$ is an open subset and $f$ is a function defined and analytic in $I$ except for poles, then $f$ is a meromorphic function on $I$.

**Examples:** rational functions $f(x) = \frac{x^2-1}{x^3+2x-1}$, Gamma function $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$, Riemann Zeta function $\zeta(s) = \sum_{k=1}^\infty k^{-s}$.

Analytic functions $\subset$ Smooth functions ($C^\infty$)
Analytic functions: more details

**Definition**

Let $I \subseteq \mathbb{R}$ be an open interval. A function $f : I \to \mathbb{R}$ is analytic at a point $x_0 \in I$ if it admits a Taylor series expansion in a neighborhood of $x_0$. If $f$ is analytic at every point of $I \subseteq \mathbb{R}$, we say that $f$ is analytic in $I$.

**Proposition**

Let $I \subseteq \mathbb{R}$ be an open interval, and let $f : I \to \mathbb{R}$ be an analytic function on $I$, then either

1. $f \equiv 0$ in $I$, or
2. the zeros of $f$ in $I$ are isolated.
Non-analytic functions: Illustrative example

The function \( f(x) \) defined by

\[
f(x) = \begin{cases} 
sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\
0, & \text{if } x = 0
\end{cases}
\]

is not analytic because the point \( x = 0 \) is a point of accumulation for the zeros of \( f \).
A **ring** is a set \( R \) together with two binary operators \(+\) and \(\ast\) satisfying conditions:

1. **Additive associativity**: For all \( a, b, c \in R \), \((a + b) + c = a + (b + c)\);
2. **Additive commutativity**: For all \( a, b \in R \), \(a + b = b + a\);
3. **Additive identity**: There exists an element \(0 \in R\) such that for all \( a \in R\), \(0 + a = a + 0 = a\);
4. **Additive inverse**: For every \( a \in R \) there exists \(-a \in R\) such that \(a + (-a) = (-a) + a = 0\);
5. **Left and right distributivity**: For all \( a, b, c \in R \), \(a \ast (b + c) = (a \ast b) + (a \ast c)\) and \((b + c) \ast a = (b \ast a) + (c \ast a)\);
6. **Multiplicative associativity**: For all \( a, b, c \in R \), \((a \ast b) \ast c = a \ast (b \ast c)\) (a ring satisfying this property is sometimes explicitly termed an **associative ring**);
7. **Multiplicative commutativity**: For all \( a, b \in R \), \(a \ast b = b \ast a\) (a ring satisfying this property is termed a **commutative ring**);
8. **Multiplicative identity**: There exists an element \(1 \in R\) such that for all \( a \neq 0 \in R \), \(1 \ast a = a \ast 1 = a\) (a ring satisfying this property is termed a **unit ring**, or sometimes a **ring with identity**);
9. **Multiplicative inverse**: For each \( a \neq 0 \in R \), there exists an element \(a^{-1} \in R\) such that for all \( a \neq 0 \in R \), \(a \ast a^{-1} = a^{-1} \ast a = 1\), where \(1\) is the identity element.
## Basic algebraic structures: Summary

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Control system
A car example

Control system

reference
desired output

input
gas pedal/wheel/etc.

output
distance/speed/etc.

controller

system

\( v(t) \quad u(t) \quad y(t) \)

closed loop
Input-output and state-space forms: single-input single-output systems

Notation: the first- and second-order derivatives are $\dot{\xi} := \frac{d\xi}{dt}$, $\ddot{\xi} := \frac{d^2\xi}{dt^2}$, and $\xi^{(k)} := \frac{d^k\xi}{dt^k}$ stands to the time derivative of an arbitrary order.

Input-output equation

$$y^{(n)} = \phi \left(y, \dot{y}, \ldots, y^{(n-1)}, u, \dot{u}, \ldots, u^{(s)}\right).$$

State equations

$$\begin{align*}
\dot{x} &= f(x, u) \\
y &= h(x),
\end{align*}$$

$x \in \mathbb{R}^n$ is the vector of state variables, $u \in \mathbb{R}$ is the input signals, $y \in \mathbb{R}$ is the output signal, $f$ and $h$ are meromorphic functions.
Let $\mathcal{R}$ denote the ring of analytic functions in a finite number of variables from the set a finite number of independent system variables from the infinite set

$$C_{ss} = \{x_i, \ i = 1, \ldots, n; \ u^{(k)}, \ k \geq 0\}$$

or

$$C_{io} = \{y, y^{(1)}, \ldots, y^{(n-1)}, \ u^{(k)}, \ k \geq 0\}.$$

$C_{ss}$ is associated to the state-space form

$C_{io}$ is associated to the input-output description
Define a time derivative operator \( \frac{d}{dt} : \mathcal{R} \to \mathcal{R} \) as

\[
\frac{d}{dt} x = f(x, u), \quad \frac{d}{dt} u_j^{(k)} = u_j^{(k+1)},
\]

\[
\frac{d}{dt} \zeta(x, u^{(k)}) = \sum_{i=1}^{n} \frac{\partial \zeta}{\partial x_i} \frac{d}{dt} x_i + \sum_{k \geq 0} \frac{\partial \zeta}{\partial u^{(k)}} \frac{d}{dt} u^{(k)},
\]

or as

\[
\frac{d}{dt} y^{(n-1)} = \phi(\cdot), \quad \frac{d}{dt} y^{(l)} = y^{(l+1)}, \text{ for } l = 0, \ldots, n - 2,
\]

\[
\frac{d}{dt} u^{(k)} = u^{(k+1)},
\]

\[
\frac{d}{dt} \xi(y^{(l)}, u^{(k)}) = \sum_{l=0}^{n-1} \frac{\partial \xi}{\partial y^{(l)}} \frac{d}{dt} y^{(l)} + \sum_{k \geq 0} \frac{\partial \xi}{\partial u^{(k)}} \frac{d}{dt} u^{(k)}.
\]

The pair \((\mathcal{R}, d/dt)\) forms an algebraic structure known as a **differential ring**.
A ring $D$ is called an *integral domain* if it does not contain any zero divisors.

It means that if $a$ and $b$ are two elements of $D$ such that $ab = 0$, then either $a = 0$ or $b = 0$ or both.

The ring $\mathcal{R}$ of analytic functions is *integral domain*. 
Remark: $C^\infty$ functions too form a ring, but it contains zero divisors.

Example: Consider two smooth functions defined as

$$f_1(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{if } x < 0, \\ 0, & \text{if } x \geq 0 \end{cases}$$

and

$$f_2(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ e^{-\frac{1}{x^2}}, & \text{if } x > 0, \end{cases}$$

whose product is identically zero.
Field of meromorphic functions $\mathcal{K}$
Construction: sketch

Construction:

1. Let $S$ be multiplicative subset of $\mathcal{R}$.
2. Consider the set of fractions of $\mathcal{R}$ over $S$, denoted as $\mathcal{K} := S^{-1}\mathcal{R}$.
3. Elements of $\mathcal{K}$ are meromorphic functions of the form $\beta^{-1}\alpha$, where $\alpha \in \mathcal{R}$, $\beta \in S$.
4. Since $\mathcal{R}$ is integral domain, $\mathcal{K}$ forms an algebraic structure known as a field of fractions (quotient field).

General idea: The field of fractions $\mathcal{K}$ of an integral domain $\mathcal{R}$ is the smallest field containing $\mathcal{R}$, since it is obtained from $\mathcal{R}$ by adding the least needed to make $\mathcal{R}$ a field, namely the possibility of dividing by any nonzero element.
The operator $d/dt$ can be extended so that $d/dt : \mathcal{K} \to \mathcal{K}$. For $b^{-1}a \in \mathcal{K}$ we define
\[
\frac{d}{dt} (b^{-1}a) := (b^2)^{-1}(\dot{a}b - a\dot{b}), \ a \in \mathcal{R}, \ b \in \mathcal{S}.
\]

The pair $(\mathcal{K}, d/dt)$ is a **differential field**.
Consider next the infinite set of symbols

\[ \text{d}C_{ss} = \{ \text{d}x_i, \ i = 1, \ldots, n; \ \text{d}u^{(k)}, \ k \geq 0 \} \]

or

\[ \text{d}C_{io} = \{ \text{d}y, \text{d}y^{(1)}, \ldots, \text{d}y^{(n-1)} i = 1, \ldots, n; \ \text{d}u^{(k)}, \ k \geq 0 \} \]

and denote by \( E \) the \textbf{differential vector space} spanned over the field \( \mathcal{K} \) by the elements of \( \text{d}C \), i.e.

\[ E := \text{span}_\mathcal{K}\{\text{d}C\}. \]
Any element of $\mathcal{E}$ has the form

$$\omega = \sum_{i=1}^{n} \alpha_i dx_i + \sum_{k \geq 0} \beta_k du^{(k)}$$

or

$$\omega = \sum_{i=1}^{n} \alpha_i dy^{(i)} + \sum_{k \geq 0} \beta_k du^{(k)},$$

where $\alpha_i, \beta_k \in \mathcal{K}$ and only a finite number of coefficients $\beta_k$ are nonzero.

The elements of $\mathcal{E}$ are called the differential **one-forms**.
Differential forms

Operators $d$ and $d/dt$ in $\mathcal{E}$

The differential operator $d : \mathcal{K} \to \mathcal{E}$ is defined as

$$d\zeta \left( x, u^{(k)} \right) = \sum_{i=1}^{n} \frac{\partial \zeta}{\partial x_i} dx_i + \sum_{k \geq 0} \frac{\partial \zeta}{\partial u^{(k)}} du^{(k)}$$

or

$$d\xi \left( y^{(l)}, u^{(k)} \right) = \sum_{l=0}^{n-1} \frac{\partial \xi}{\partial y^{(l)}} dy^{(l)} + \sum_{k \geq 0} \frac{\partial \xi}{\partial u^{(k)}} du^{(k)}.$$

For the one-form $\omega = \lambda_i d\varphi_i$, where $\lambda_i \in \mathcal{K}$ and $\varphi_i \in \mathcal{C}$, the operator $d/dt : \mathcal{E} \to \mathcal{E}$ is defined as

$$\frac{d}{dt} \left( \sum_{l} \lambda_l d\varphi_l \right) := \sum_{l} \left( \dot{\lambda}_l d\varphi_l + \lambda_l d\dot{\varphi}_l \right).$$

Remark: Operators $d$ and $d/dt$ commute, i.e. for $\varphi \in \mathcal{K}$

$$\frac{d}{dt}(d\varphi) = d \left( \frac{d}{dt} \varphi \right) = d\dot{\varphi}.$$
**Differential forms**

**Example:** Let $F = \sin(x_1 x_2) \in \mathcal{K}$. Then differentiating $F$ with respect to $x_1$ and $x_2$, we get

$$dF = \cos(x_1 x_2)x_2 dx_1 + \cos(x_1 x_2)x_1 dx_2 = \cos(x_1 x_2)[x_2 dx_1 + x_1 dx_2]$$

with $dF \in \mathcal{E}$. 
Starting from the space $\mathcal{E}$ it is possible to build up the structures used in **exterior differential calculus**. Define the set

$$\wedge d\mathcal{C} = \{ d\zeta \wedge d\eta \mid \zeta, \eta \in \mathcal{C} \},$$

where $\wedge$ denotes the **wedge product** with the standard properties

$$d\zeta \wedge d\eta = -d\eta \wedge d\zeta \quad \text{and} \quad d\zeta \wedge d\zeta = 0$$

for $\zeta, \eta \in \mathcal{C}$.

Introduce the space $\mathcal{E}^2 = \text{span}_K \wedge d\mathcal{C}$ with elements being **two-forms**. The operator $d : \mathcal{E} \to \mathcal{E}^2$, called **exterior derivative** operator, is defined for $\omega = \sum_{\ell=1}^{k} \alpha_{\ell}(\zeta_1, \ldots, \zeta_k) d\zeta_\ell \in \mathcal{E}$, where $\zeta_1, \ldots, \zeta_k \in \mathcal{C}$, by the rule

$$d\omega := \sum_{\ell, \ell'} \frac{\partial \alpha_{\ell}}{\partial \zeta_{\ell'}} d\zeta_{\ell} \wedge d\zeta_{\ell'}.$$
Example: Let $\omega = dx_1 - \frac{x_1}{x_2} dx_2$, then

$$d\omega = d[dx_1 - \frac{x_1}{x_2} dx_2] = d[dx_1] - d \left[ \frac{x_1}{x_2} dx_2 \right]$$

$$= - \frac{\partial}{\partial x_1} \left( \frac{x_1}{x_2} \right) dx_1 \wedge dx_2 - \frac{\partial}{\partial x_2} \left( \frac{x_1}{x_2} \right) dx_2 \wedge dx_2$$

$$= - \frac{1}{x_2} dx_1 \wedge dx_2$$

Remark: The notion of two-form can be generalized to the $s$-form and wedge product is defined for arbitrary $s$-forms.
Definition

A one-form $\omega \in \mathcal{E}$ is **closed**, if $d\omega = 0$.

Definition

A one-form $\omega \in \mathcal{E}$ is **exact**, if $\omega = d\zeta$ for some $\zeta \in \mathcal{K}$.

Proposition

*Any exact one-form is closed.*
Lemma (Poincaré’s Lemma)

Let \( \omega \) be a closed one-form in \( \mathcal{E} \). Then there exists \( \varphi \in \mathcal{K} \) such that locally \( \omega = d\varphi \).

**Example:** Consider a closed one-form

\[
\omega = \frac{x_2}{x_1^2 + x_2^2} \, dx_1 - \frac{x_1}{x_1^2 + x_2^2} \, dx_2.
\]

Locally around points

- \((x_1, x_2)\) such that \( x_2 \neq 0 \), we get \( \omega = d[\arctan(x_1/x_2)] \);
- \((x_1, x_2)\) such that \( x_1 \neq 0 \) and \( x_2 = 0 \), we get \( \omega = d[\arctan(-x_2/x_1)] \).

However, there is no function \( \varphi \) such that \( \omega = d\varphi \) globally.
Frobenius theorem

Definition

A subspace $\Omega \subset \mathcal{E}$ is closed or integrable, if $\Omega$ has a basis which consists only of closed forms.

Theorem

Let $\Omega = \text{span}_K \{\omega_1, \ldots, \omega_\kappa\}$. The subspace $\Omega$ is integrable if and only if

$$d\omega_i \wedge \omega_1 \wedge \cdots \wedge \omega_\kappa = 0$$

for all $i = 1, \ldots, \kappa$. 
Example: Consider the one-form $\omega = dx_1 + x_1 dx_2$. To verify whether $\omega$ is closed or not we need to find the exterior derivative as

$$d\omega = d[dx_1 + x_1 dx_2] = d[dx_1] + d[x_1 dx_2] = 0$$

$$= \frac{\partial x_1}{\partial x_1} dx_1 \wedge dx_2 + \frac{\partial x_1}{\partial x_2} dx_2 \wedge dx_2 = dx_1 \wedge dx_2.$$

Therefore, $\omega$ is not closed since $d\omega \neq 0$.

However, the vector space $\text{span}_K\{\omega\}$ is integrable since

$$d\omega \wedge \omega = dx_1 \wedge dx_2 \wedge (dx_1 + x_1 dx_2)$$

$$= dx_1 \wedge dx_2 \wedge dx_1 + x_1 dx_1 \wedge dx_2 \wedge dx_2 = 0.$$

Finally, if we choose the integrating factor $\alpha = 1/x_1$, then $\omega$ becomes integrable and $F = \ln |x_1| + x_2$. 

Frobenius theorem
Sequence $\mathcal{H}_k$

A sequence of subspaces

$\mathcal{H}_0 \supset \cdots \supset \mathcal{H}_k^* \supset \mathcal{H}_{k^*+1} = \mathcal{H}_{k^*+2} = \cdots =: \mathcal{H}_\infty$ of $\mathcal{E}$ is defined by

$$
\begin{align*}
\mathcal{H}_0 &= \text{span}_K \{ dx_1, \ldots, dx_n, du \}, \\
\mathcal{H}_k &= \{ \omega \in \mathcal{H}_{k-1} \mid \dot{\omega} \in \mathcal{H}_{k-1} \}, \quad k \geq 1,
\end{align*}
$$

or

$$
\begin{align*}
\mathcal{H}_1 &= \text{span}_K \left\{ dy, \ldots, dy^{(n-1)}, du, \ldots, du^{(s)} \right\}, \\
\mathcal{H}_{k+1} &= \{ \omega \in \mathcal{H}_k \mid \dot{\omega} \in \mathcal{H}_k \}, \quad k \geq 1.
\end{align*}
$$

Sequence $\mathcal{H}_k$ plays an important role in the analysis of the structural properties of nonlinear systems.
A **skew polynomial ring** \( A[\partial; \alpha, \beta] \) is a noncommutative polynomial ring in \( \partial \) with coefficients in \( A \) satisfying

\[
\forall a \in A, \quad \partial a = \alpha(a)\partial + \beta(a).
\]

Each polynomial \( \pi \in A[\partial; \alpha, \beta] \) can be uniquely written in the form

\[
\pi = \sum_{\ell=0}^{k} \pi_{\ell} \partial^{k-\ell}, \quad \pi_{\ell} \in A.
\]

If \( \pi_{0} \neq 0 \), then \( k \) is called the degree of \( \pi \), denoted by \( \text{deg}(\pi) \).
Several special cases:

- **Ring of differential operators**: \( \mathcal{A}[\partial; \text{id}, \frac{d}{dt}] \).
- Ring of shift operators: \( \mathcal{A}[\partial; \sigma, 0] \), \( \mathcal{A}[\partial; \delta, 0] \).
- Ring of difference operators: \( \mathcal{A}[\partial; \tau, \tau - \text{id}] \) with \( \tau a(x) = a(x + 1) \).

**Definition**

The skew polynomial ring, induced by \((\mathcal{K}, d/dt)\), is the ring \( \mathcal{K}[\partial; \text{id}_\mathcal{K}, d/dt] := \mathcal{K}[\partial; d/dt] \) of polynomials with usual addition and multiplication satisfying, for any \( \varsigma \in \mathcal{K} \subset \mathcal{K}[\partial; d/dt] \), the commutation rule

\[
\partial \varsigma := \varsigma \partial + \dot{\varsigma}.
\]
Example 1: Consider multiplication of two polynomials \( p(\partial) = \partial^2 + 1 \) and \( q(\partial) = y\partial - 1 \)

\[
p(\partial)q(\partial) = (\partial^2 + 1)(y\partial - 1) = \partial^2 y\partial - \partial^2 + y\partial - 1 \\
= \partial(y\partial^2 + \dot{y}\partial) - \partial^2 + y\partial - 1 \\
= y\partial^3 + \dot{y}\partial^2 + \dot{y}\partial^2 + \ddot{y}\partial - \partial^2 + y\partial - 1 \\
= y\partial^3 + (2\dot{y} - 1)\partial^2 + (\ddot{y} + y)\partial - 1
\]

Example 2:

\[
\partial \cdot (y + u + 1) = y\partial + \dot{y} + u\partial + \dot{u}, \\
(y + u + 1) \cdot \partial = y\partial + u\partial + \partial.
\]
Recall that $\mathcal{K}[\partial; d/dt]$ is the skew polynomial ring, where $\partial$ is a polynomial indeterminate. Multiplication in $\mathcal{K}[\partial; d/dt]$ is defined by the commutation rule $\partial \varsigma = \varsigma \partial + \dot{\varsigma}$, $\alpha \in \mathcal{K}$.

Polynomial system description

\[ y^{(n)} = \phi \left( y, \dot{y}, \ldots, y^{(n-1)}, u, \dot{u}, \ldots, u^{(s)} \right) \]

\[ \partial^i dy := dy^{(i)} \]
\[ \partial^j du := du^{(j)} \]

\[ \partial^n - \sum_{i=0}^{n-1} p_i \partial^i \] \[ dy - \sum_{j=0}^{s} q_j \partial^j du = 0 \]

\[ p(\partial)dy + q(\partial)du = 0 \]
Consider the nonlinear system

\[ \ddot{y} = \dot{u}y + u^2 \dot{y}. \]

Define \( \phi := \dot{u}y + u^2 \dot{y} \) and differentiate it with respect to \( y, \dot{y}, u \) and \( \dot{u} \)

\[
p_0 = \frac{\partial \phi}{\partial y} = \dot{u}, \quad p_1 = \frac{\partial \phi}{\partial \dot{y}} = u^2, \]
\[
q_0 = \frac{\partial \phi}{\partial u} = 2u \dot{y}, \quad q_1 = \frac{\partial \phi}{\partial \dot{u}} = y. \]

Using relations \( \partial^i dy := dy^{(i)} \) and \( \partial^j du := du^{(j)} \), we get

\[
(\partial^2 - u \partial - \dot{u}) dy - (y \partial + 2u \dot{y}) du = 0. \]
**Algebraic and polynomial formalism: Summary**

**Actual picture**

- $\mathcal{C}$ – infinite set of system variables
- $\mathcal{K}$ – differential field of meromorphic functions
- $\mathcal{E} = \text{span}_\mathcal{K}\{d\mathcal{C}\}$ – vector space of differential one-forms
- $\mathcal{E}^{(s)} = \text{span}_\mathcal{K}\{\wedge^s d\mathcal{C}\}$ – vector space of differential $s$-forms
- $\mathcal{K}[\partial; \frac{d}{dt}]$ – skew polynomial ring
**Problem statement**

\[ y^{(n)} = \phi \left( y, \dot{y}, \ldots, y^{(n-1)}, u, \dot{u}, \ldots, u^{(s)} \right) \]

\[ \dot{x} = f(x, u) \]
\[ y = h(x) \]

**Goal:**

Find, if possible, the state coordinates \( x(t) \in \mathbb{R}^n \) such that in these coordinates the system takes the minimal state-space form.

**Definition**

The state-space description is said to be realization of the i/o equation if both equations have the same solution sets \( \{(u(t), y(t)), t \geq 0\} \).
Some of the existing results are based:

- on the sequence of distributions of vector fields
  A. J. van der Shaft, 1987
- on the iterative Lie brackets of the vector fields
  E. Delaleau and W. Respondek, 1995
- on the sequence of the subspaces of differential one-forms
  G. Conte, C. H. Moog, and A. M. Perdon, 2007
- on polynomial framework
  Ü. Kotta, M. Tönso, and J. Belikov, 2009-...

Polynomial approach:

- System is described by two polynomials from the skew polynomial ring
- Solution in terms of polynomials ⇒ explicit formulas
- More transparent and simple ⇒ easy to implement in symbolic software
- Similar to the linear case ⇒ easier to understand
Recall that the sequence of subspaces \( \{\mathcal{H}_k\}_{k=1}^{\infty} \) of \( E \) is defined as

\[
\mathcal{H}_1 = \text{span}_K \left\{ dy, \ldots, dy^{(n-1)}, du, \ldots, du^{(s)} \right\},
\]
\[
\mathcal{H}_{k+1} = \{ \omega \in \mathcal{H}_k \mid \dot{\omega} \in \mathcal{H}_k \}, \quad k \geq 1.
\]

**Theorem**

The nonlinear i/o equation has an observable state-space realization if and only if the subspace \( \mathcal{H}_{s+2} \) is integrable.

**Corollary**

The state coordinates can be obtained by integrating the exact basis vectors of \( \mathcal{H}_{s+2} \).
Realization algorithm: general idea

1. **i/o equations**
2. **Assumptions**
   - True: Define polynomial ring
   - False: Stop
3. **Find** $P(\partial)$ and $Q(\partial)$
4. **Compute one-forms: $\omega_l$**
5. **System is not realizable**
   - False: Stop
   - True: $\mathcal{H}_{s+2}$ is closed

- **Integrate one-forms**
- **Compute state equations**
  - State-space form
Computation of $\mathcal{H}_{s+2}$: polynomial method

Subspace $\mathcal{H}_{s+2}$ can be calculated as

$$\mathcal{H}_{s+2} = \text{span}_K \{\omega_1, \ldots, \omega_n\},$$

where

$$\omega_l = \begin{bmatrix} p_l(\partial) & q_l(\partial) \end{bmatrix} \begin{bmatrix} dy \\ du \end{bmatrix},$$

for $l = 1, \ldots, n$, and $p_l(\partial)$ and $q_l(\partial)$ can be recursively calculated from the equalities

$$p_{l-1}(\partial) = \partial p_l(\partial) + \xi_l, \quad \text{deg} \, \xi_l = 0,$$
$$q_{l-1}(\partial) = \partial q_l(\partial) + \gamma_l, \quad \text{deg} \, \gamma_l = 0$$

with the initial polynomials $p_0(\partial) := p(\partial)$ and $q_0(\partial) := q(\partial)$. 
Consider the nonlinear i/o system

\[ \ddot{y} = \dot{u}\dot{y} + uy \]

that can be described by two polynomials

\[ p(\partial) = \partial^2 - \dot{u}\partial - u \quad \text{and} \quad q(\partial) = -y\partial - u. \]

Calculate two sequences of the left quotients as: \( p_1(\partial) = \partial - \dot{u}, p_2 = 1, \) and \( q_1(\partial) = -\dot{y}, q_2 = 0. \) Then, the one-forms are

\[ \omega_1 = p_1(\partial)dy + q_1(\partial)du = (\partial - \dot{u})dy - \dot{y}du = d\dot{y} - \dot{u}dy - \dot{y}du, \]

\[ \omega_2 = p_2(\partial)dy + q_2(\partial)du = dy, \]

and the subspace \( \mathcal{H}_{s+2} = \mathcal{H}_3 = \text{span}_K\{dy, d\dot{y} - \dot{y}du\} \) is integrable. The choice \( x_1 = y, x_2 = e^{-u}\dot{y} \) yields the state equations

\[ \dot{x}_1 = e^u x_2 \]

\[ \dot{x}_2 = e^{-u}ux_1 \]

\[ y = x_1. \]
Consider the "ball and beam" system
\[
\ddot{y} = \frac{mR^2}{J + mR^2} \left( y \dot{u}^2 - g \sin(u) \right),
\]
where \( J, R, m, g \) are some physical parameters. The i/o equation can be described in polynomial form as
\[
\begin{align*}
p(\partial) &= \partial^2 - \frac{mR^2 \dot{u}^2}{J + mR^2} \quad \text{and} \quad q(\partial) = -\frac{2mR^2 y \dot{u}}{J + mR^2} \partial + \frac{gmR^2 \cos(u)}{J + mR^2}.
\end{align*}
\]
Compute the left quotients as: \( p_1(\partial) = \partial, p_2(\partial) = 1 \) and \( q_1(\partial) = -\frac{2mR^2}{J + mR^2} y \dot{u}, q_2(\partial) = 0 \). Then, we get
\[
\mathcal{H}_3 = \text{span}_K \{\omega_1, \omega_2\} = \text{span}_K \{dy, d\dot{y} - \frac{2mR^2}{J + mR^2} y \dot{u} du\},
\]
which by the Frobenius theorem is not closed, since
\[
d\omega_2 \wedge \omega_1 \wedge \omega_2 = \frac{2mR^2}{J + mR^2} y \dot{u} du \wedge d\dot{u} \wedge dy \wedge d\dot{y} \neq 0.
\]
Therefore, the i/o equation does not admit the minimal state-space realization.
Realization: open problems

**Remark:**

- The realizability conditions are constructive and can be checked using $\mathcal{H}_{s+2}$.
- To find the state coordinates, one has to integrate the differential one-forms. The integration of (integrable in principle) differential one-forms is known to be a difficult task, in general.
- Theorem (realizability) does not define explicitly the class of i/o equations that have state-space form.

Therefore, the alternative way to tackle the realization problem is to **single out** the realizable structures for low-order i/o equations as well as to understand what can happen in case of arbitrary order, suggesting some **subclasses** of general order.
Consider the second-order i/o equation

\[ \ddot{y} = \phi(y, \dot{y}, u, \dot{u}) \]

that can be described by two polynomials

\[ p(\partial) = \partial^2 - \frac{\partial \phi}{\partial \dot{y}} \partial - \frac{\partial \phi}{\partial y} \]

and

\[ q(\partial) = -\frac{\partial \phi}{\partial \dot{u}} \partial - \frac{\partial \phi}{\partial u}. \]
Since $s = 1$, we have to check the integrability of the subspace $\mathcal{H}_3 = \text{span}_K\{\omega_1, \omega_2\}$, where

$$\omega_1 = p_1(\partial)dy + q_1(\partial)du = \dot{y} - \frac{\partial \phi}{\partial \dot{u}} du,$$
$$\omega_2 = p_2(\partial)dy + q_2(\partial)du = dy.$$

The integrability can be checked using the Frobenius theorem, i.e. to check

$$d\omega_i \wedge \omega_1 \wedge \cdots \wedge \omega_\kappa = 0$$

for all $i = 1, \ldots, \kappa$.

The first condition $d\omega_2 \wedge \omega_1 \wedge \omega_2 = 0$ is trivially satisfied.
The second condition $d\omega_1 \wedge \omega_1 \wedge \omega_2 = 0$ can be represented as

$$d \left[ d\dot{y} - \frac{\partial \phi}{\partial \dot{u}} du \right] \wedge \left[ d\dot{y} - \frac{\partial \phi}{\partial \dot{u}} du \right] \wedge dy = 0$$

or

$$\left[ -\frac{\partial^2 \phi}{\partial \dot{u} \partial y} dy \wedge du - \frac{\partial^2 \phi}{\partial \dot{u} \partial \dot{y}} \dot{y} \wedge du - \frac{\partial^2 \phi}{\partial \dot{u} \partial \dot{u}} d\dot{u} \wedge du \right. 
- \left. \frac{\partial^2 \phi}{\partial \dot{u} \partial \dot{u}} d\dot{u} \wedge du \right] \wedge \left[ d\dot{y} - \frac{\partial \phi}{\partial \dot{u}} du \right] \wedge dy = 0.$$
Realization: open problems

Using the basic properties of the exterior product $d\zeta \wedge d\zeta = 0$ and $d\varepsilon \wedge d\eta = -d\eta \wedge d\varepsilon$, the above condition can be simplified as

$$-\frac{\partial^2 \phi}{\partial \dot{u} \partial \dot{u}} du \wedge d\dot{u} \wedge dy \wedge d\dot{y} = 0.$$ 

From the above equation, we get the partial differential equation

$$\frac{\partial^2 \phi}{\partial \dot{u} \partial \dot{u}} = 0.$$ 

The solutions of equation the obtained PDE define the complete subclass of the second-order i/o equations to be realizable in the state-space form. One particular solution is:

$$\phi = \phi_1(y, \dot{y}, u) + \phi_2(y, \dot{y}, u) \dot{u}.$$
Consider the third-order i/o equation

$$y^{(3)} = \phi(y, \dot{y}, \ddot{y}, u, \dot{u}, \ddot{u}).$$

Proceeding in the same manner as in case of the second-order i/o equation, we get the system of partial differential equations

$$\begin{cases}
\phi \dddot{u} = 0 \\
\phi \dddot{u} + \phi \dddot{u} \phi \dddot{y} = 0 \\
\phi \dddot{u} - \phi \dddot{u} + 2\phi \dddot{u} \phi \dddot{y} - \phi \dddot{u} \phi \dddot{y} - (\phi \dot{u} + \phi \phi \dot{y}) \phi \dddot{y} + \phi \phi \dddot{u} \phi \dddot{y} = 0,
\end{cases}$$

where $\phi_{\alpha\beta} = \frac{\partial^2 \phi}{\partial \beta \partial \alpha}$ is used to denote the partial derivative of a function.