# ISS0031 Modeling and Identification

Lecture 8

## Introduction

**Game theory** is the study of strategic decision making. We all know many entertaining games, such as chess, poker, tic-tac-toe, bridge, baseball, computer games, etc. In addition, there is a vast area of economic games and the related political games. The competition between firms, the conflict between management and labor, the fight to get bills through congress, the power of the judiciary, war and peace negotiations between countries, and so on, all provide examples of games in action. There are also psychological games played on a personal level, where the weapons are words, and the payoffs are good or bad feelings. There are biological games, the competition between species, where natural selection can be modeled as a game played between genes. There is a connection between game theory and the mathematical areas of logic and computer science. One may view theoretical statistics as a two person game in which nature takes the role of one of the players.

The earliest example of a formal game-theoretic analysis is the study of a duopoly by A. Cournot in 1838. The mathematician E. Borel suggested a formal theory of games in 1921, which was furthered by John von Neuman in 1928 in a "theory of parlor games". Game theory was established as a field in its own right after the 1944 publication of the monumental volume *Theory of Games and Economic Behavior* by von Neuman and the economist O. Morgenstern.

## Game theory: basic notions

The essential elements of a game are **players**, **actions**, **payoffs**, **information**, etc. These are collectively known as the rules of the game, and the modeler's objective is to describe a situation in terms of the **rules of a game** so as to explain what will happen in that situation. Trying to maximize their payoffs, the players will devise plans known as strategies that pick actions depending on the information that has arrived at each moment. The combination of strategies chosen by each player is known as the equilibrium. Given an equilibrium, the modeler can see what actions come out of the conjunction of all the players' plans, and this tells him the **outcome** of the game.

**Definition 1.** ***Players*** *are the individuals who make decisions. Each player's goal is to maximize his utility by choice of actions.*

There is the **number of players** which will be denoted by $n$. Let us label the players with the integers 1 to $n$, and denote the **set of players** by $N = 1, 2, \ldots, n$. Further we study only two person games, $n = 2$, where the concepts are clearer and the conclusions are more definite. When specialized to one-player, the theory is simply called decision theory. Games of solitaire and puzzles are examples of one-person games as are various sequential optimization problems found in operations research, optimization, linear programming, or gambling. In macroeconomic models, the number of players can be very large, ranging into the millions. In such models it is often preferable to assume that there are an infinite number of players. In fact it has been found useful in many situations to assume there are a continuum of players, with each player having an infinitesimal influence on the outcome.

**Remark 1.** ***Nature*** *is a pseudo-player who takes random actions at specified points in the game with specified probabilities.*

**Definition 2.** *An* ***action*** *or* ***move*** *by player i is a choice he can make.*

By player's **payoff** we mean either:

- The utility player receives after all players and nature have picked their strategies and the game has been played out; or
- The expected utility he receives as a function of the strategies chosen by himself and the other players.

**Definition 3.** *Player's* ***strategy*** *is a rule that tells him which action to choose at each instant of the game, given his information set.*

**Definition 4.** *A* ***pure strategy*** *provides a complete definition of how a player will play a game. In particular, it determines the move a player will make for any situation he or she could face. A player's strategy set is the set of pure strategies available to that player.*

**Definition 5.** *A* ***mixed strategy*** *is an assignment of a probability to each pure strategy. This allows for a player to randomly select a pure strategy. Since probabilities are continuous, there are infinitely many mixed strategies available to a player, even if their strategy set is finite.*

Of course, one can regard a pure strategy as a degenerate case of a mixed strategy, in which that particular pure strategy is selected with probability 1 and every other strategy with probability 0. A pure strategy constitutes a rule that tells the player what action to choose, while a mixed strategy constitutes a rule that tells him what dice to throw in order to choose an action. If a player pursues a mixed strategy, he might choose any of several different actions in a given situation, an unpredictability which can be helpful to him. Mixed strategies occur frequently in the real world. In American football games, for example, the offensive team has to decide whether to pass or to run. Passing generally gains more yards, but what is most important is to choose an action not expected by the other team. Teams decide to run part of the time and pass part of the time in a way that seems random to observers but rational to game theorists.

**Definition 6.** ***Conflict situation*** *exists when two or more decision makers who have different objectives act on the same system or share the same resources. Game theory studies mathematical models of conflict and cooperation between intelligent rational decision-makers.*

**Types of games:**

- A **cooperative** game is a game in which the players can make binding commitments, as opposed to a **noncooperative** game, in which they cannot.
- When the players know all past moves by all the players and the outcomes of all past random moves, the game is said to be of **perfect information**; otherwise, **imperfect information**. Most games studied in game theory are imperfect-information games. Recreational games of perfect information games include chess, go, and mancala. Many card games are games of imperfect information, for instance poker or contract bridge.
- Two-person games of perfect information with win or lose outcome and no chance moves are known as **combinatorial games**.
- A **zero-sum** game is a game in which the sum of the payoffs of all the players is zero whatever strategies they choose. A game which is not zero-sum is **nonzero-sum** game or **variable-sum**.

**Definition 7.** *The **outcome** of the game is a set of interesting elements that the modeler picks from the values of actions, payoffs, and other variables after the game is played out.*

**Remark 2.** *Next, we consider only zero-sum games with two players $A$ and $B$.*

## Game matrix

A **game matrix** or **payoff matrix** is a table that shows all of the possible moves of each player in a game, and the payoffs or rewards that each player will receive from all possible strategies. From a game matrix, one can visualize a given player's strategies, and make predictions of the outcome of a game. Since the game is zero-sum, win of the one player means lose of the other player.

Suppose that player $A$ has $m$ strategies $A_1, A_2, \ldots, A_m$ and player $B$ has $n$ strategies $B_1, B_2, \ldots, B_n$, then by the game matrix we mean the table

| $A$ | $B$ | | | |
|---|---|---|---|---|
| | $B_1$ | $B_2$ | $\ldots$ | $B_n$ |
| $A_1$ | $a_{11}$ | $a_{12}$ | $\ldots$ | $a_{1n}$ |
| $A_2$ | $a_{21}$ | $a_{22}$ | $\ldots$ | $a_{2n}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $A_m$ | $a_{m1}$ | $a_{m2}$ | $\ldots$ | $a_{mn}$ |

In this form, player $A$ chooses a row, player $B$ chooses a column, and $B$ pays $A$ the entry in the chosen row and column. Note that the entries of the matrix are the winnings of the row chooser and losses of the column chooser.

**Example 1:** Peter and Paul decide to play the following game: each one of them simultaneously indicates a number with the fingers of his right hand. If the two numbers are both even, or if they are both odd, Peter gives Paul 2 shillings. If Peter's number is even and Paul's is odd, the latter gives Peter 1 shilling.

Such a game can be represented with the help of the following tableau, which shows how much Peter wins in different situations.

| Peter | Paul | |
|---|---|---|
| | Even | Odd |
| Even | $-2$ | 3 |
| Odd | 1 | $-2$ |

**Example 2:** "Where is the coin?"

Suppose that:

- $A$ is the player, who has a coin. Then, his strategies are as follows: $A_1$ – coin is in the left hand, $A_2$ – coin is in the right hand;
- $B$ is the guessing player. Then, his strategies are as follows: $B_1$ – coin is in the left hand, $B_2$ – coin is in the right hand.

The positive result (getting the coin) of the game is denoted by 1 and the negative by $-1$. Then, the game matrix can be represented in the following form

| $A$ | $B$ | |
|---|---|---|
| | $B_1$ | $B_2$ |
| $A_1$ | $-1$ | 1 |
| $A_2$ | 1 | $-1$ |

**Example 3:** Rock-scissors-paper game.

Rock-scissors-paper is a hand game played by two people. The basic idea of the game is the following: both players show simultaneously one of three figures. Rules of the game are such that: paper beats rock, rock beats scissors, and scissors beats paper. Let game strategies are: $A_1, B_1$ show rock, $A_2, B_2$ show scissors, $A_3, B_3$ show paper. If figures have the same name, then payoff is 0; otherwise $\pm 1$.

| $A$ | $B$ | | |
|---|---|---|---|
| | $B_1$ | $B_2$ | $B_3$ |
| $A_1$ | 0 | 1 | $-1$ |
| $A_2$ | $-1$ | 0 | 1 |
| $A_3$ | 1 | $-1$ | 0 |

## Solution of matrix games

The result of the game is always reward of one player and payoff of another player. Usually, the outcome of the game can be achieved by the infinite number of steps. Thus, the probability has to be involved. In order to find the outcome, all strategies have to be used with certain probability.

A mixed strategy for player $A$ may be represented by an $m$-tuple, $P = (p_1, p_2, \dots, p_m)$ of probabilities that add to 1, i.e. $p_1 + p_2 + \cdots + p_m = 1$. If $A$ uses the mixed strategy $P = (p_1, p_2, \dots, p_m)$ and $B$ chooses column $j$, then the (average) payoff to $A$ is $\sum_{i=1}^{m} p_i a_{ij}$. Similarly, a mixed strategy for player $B$ is an $n$-tuple $Q = (q_1, q_2, \dots, q_n)$. If $B$ uses $Q$ and $A$ uses row $i$ the payoff to $A$ is $\sum_{j=1}^{n} a_{ij} q_j$. More generally, if $A$ uses the mixed strategy $P$ and $B$ uses the mixed strategy $Q$, the (average) payoff to $A$ is the function $\nu$ expressing mathematical expectation

$$\nu = E(P, Q) = P\mathcal{A}Q^T = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j.$$

**Example 4:** Let the game is given by the matrix

$$\mathcal{A} = \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}$$

Player $A$ uses the first and second strategies with probabilities of $\frac{1}{7}$ and $\frac{6}{7}$, respectively. Player $B$ uses the first and second strategies with probabilities of $\frac{3}{7}$ and $\frac{4}{7}$, respectively. Therefore, $P = (\frac{1}{7}, \frac{6}{7})$ and $Q = (\frac{3}{7}, \frac{4}{7})$. Which of the players has advantage in the game?

$$\nu = E(P, Q) = \begin{pmatrix} \frac{1}{7} & \frac{6}{7} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} \frac{3}{7} \\ \frac{4}{7} \end{pmatrix} = \frac{53}{49}$$

Therefore, the game is clearly in $A$'s favor.

## Solution of matrix games in pure strategies

Note that it always necessary to verify whether game is solvable in pure strategies.

**Domination:**

**Definition 8.** *We say the $i$th row of a matrix $\mathcal{A} = (a_{ij})$ **dominates** the $k$th row if $a_{ij} \geq a_{kj}$ for all $j = 1, \dots, n$. We say the $i$th row of $\mathcal{A}$ **strictly dominates** the $k$th row if $a_{ij} > a_{kj}$ for all $j = 1, \dots, n$. Similarly, the $j$th column of $\mathcal{A}$ dominates (strictly dominates) the $k$th column if $a_{ij} \leq a_{ik}$ (resp. $a_{ij} < a_{ik}$) for all $i = 1, \dots, m$.*

**Example 5:** Let the game matrix is

$$\mathcal{A} = \begin{pmatrix} 4 & 5 \\ -1 & 0 \\ 5 & 5 \end{pmatrix}$$

One can see that the third row is maximal. Thus, it dominates the first and second rows; hence, we get $p_1 = p_2 = 0$ and $p_3 = 1$. We choose the minimal column, which is the first one, for player $B$; hence, $q_1 = 1$ and $q_2 = 0$. Therefore, the solution is $P = (0, 0, 1)$, $Q = (1, 0)^T$

$$\nu = E(P, Q) = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 5 \\ -1 & 0 \\ 5 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 5.$$

**Saddle point:**
If some entry $a_{ij}$ of the matrix $\mathcal{A}$ has the property that

- $\alpha_i = a_{ij}$ is the minimum of the $i$th row, i.e. $\alpha_i = \min\limits_{j=1,\ldots,n} a_{ij}$;
- $\beta_j = a_{ij}$ is the maximum of the $j$th column, i.e. $\beta_j = \max\limits_{i=1,\ldots,m} a_{ij}$,

then we say $a_{ij}$ is a saddle point. If $a_{ij}$ is a saddle point, then player $A$ can then win at least $a_{ij}$ by choosing $i$th row, and player $B$ can keep loss to at most $a_{ij}$ by choosing $j$th column. Hence, the lower and upper values of the game are:

$$\alpha = \max_{i=1,\ldots,m} \alpha_i = \max_{i=1,\ldots,m} \min_{j=1,\ldots,n} a_{ij},$$
$$\beta = \min_{j=1,\ldots,n} \beta_j = \min_{j=1,\ldots,n} \max_{i=1,\ldots,m} a_{ij}.$$

If $\alpha = \beta$, then

- game is solvable using pure strategies;
- game has a saddle point, which is $\alpha = \beta = a_{kl}$;
- optimal strategies for players $A$ and $B$ are $P^* = (0, \ldots, 0, \underbrace{1}_{p_k^*}, 0, \ldots, 0)$ and $Q^* = (0, \ldots, 0, \underbrace{1}_{q_l^*}, 0, \ldots, 0)$, respectively.

**Example 6:** Let the game is given by the following table

| $A$ | $B$ | | $\alpha_i$ |
|---|---|---|---|
| | $B_1$ | $B_2$ | |
| $A_1$ | $-1$ | $1$ | $-1$ |
| $A_2$ | $1$ | $-1$ | $-1$ |
| $\beta_j$ | $1$ | $1$ | $\alpha = -1$ <br> $\beta = 1$ |

Since $\alpha \neq \beta$, there is no saddle point.

**Example 7:** Let the game matrix is

$$\mathcal{A} = \begin{pmatrix} 4 & 8 & 3 \\ 5 & -2 & 1 \\ 1 & 17 & 0 \end{pmatrix}$$

The game matrix $\mathcal{A}$ can represented by the following table

| $A$ | $B$ | | | $\alpha_i$ |
|---|---|---|---|---|
| | $B_1$ | $B_2$ | $B_3$ | |
| $A_1$ | 4 | 8 | 3 | 3 |
| $A_2$ | 5 | $-2$ | 1 | $-2$ |
| $A_3$ | 1 | 17 | 0 | 0 |
| $\beta_j$ | 5 | 17 | 3 | $\alpha = 3$ $\beta = 3$ |

Now, one may easily see that $\alpha = \beta = 3$ and therefore the game is solvable using pure strategies. The game has a saddle point, which is $a_{13} = 3$. Therefore, the optimal strategies for players $A$ and $B$ are $P^* = (1, 0, 0)$ and $Q^* = (0, 0, 1)$, respectively.

**Remark 3.** *If there exists a saddle point, then $\alpha = \beta = \nu^*$ is the value of the game and $P^*$ is called a maximum strategy as well as $Q^*$ is called a minimum strategy.*

Note that there is a relation between two considered solution methods, namely Domination $\subset$ Saddle point.

**Example 8:** Let the game matrix is

$$\mathcal{A} = \begin{pmatrix} 4 & 5 \\ -1 & 0 \\ 5 & 5 \end{pmatrix}$$

The game matrix $\mathcal{A}$ can represented by the following table

| $A$ | $B$ | | $\alpha_i$ |
|---|---|---|---|
| | $B_1$ | $B_2$ | |
| $A_1$ | 4 | 5 | 4 |
| $A_2$ | $-1$ | 0 | $-1$ |
| $A_3$ | 5 | 5 | 5 |
| $\beta_j$ | 5 | 5 | $\alpha = 5$ $\beta = 5$ |

Thus, one may easily see that $P^* = (0, 0, 1)$ and $Q_1^* = (1, 0)$, $Q_2^* = (0, 1)$.

## Game as a linear programming problem

Recall that $\mathcal{A}$ is the game matrix, $P = (p_1, \ldots, p_m)$ are mixed strategies of player $A$, and $Q = (q_1, \ldots, q_n)$ are mixed strategies of player $B$.

**Definition 9.**

- *An optimal mixed strategy of player $A$ is a vector $P^*$ such that*

$$E(P^*, j) \geq \nu^*$$

*for each pure strategy $B_j$ ($j = 1, \ldots, n$) of player $B$.*

- *An optimal mixed strategy of player $B$ is a vector $Q^*$ such that*

$$E(Q^*, i) \leq \nu^*$$

*for each pure strategy $A_i$ ($i = 1, \ldots, m$) of player $A$.*

Number $\nu^*$ is called value of a game.

**Remark 4.** *Note that adding number $c$ to each element of the game matrix $\mathcal{A}$ does not affect the optimal strategies of players. However, the value $\nu^*$ of game increases such that the new value of the game is $\nu^* + c$. Due to this fact we can assume that elements of the matrix $\mathcal{A}$ are non-negative.*

Let us consider the game problem from player $A$'s point of view. If player $B$ uses strategy $B_j$, then the reward of player $A$ is

$$E(P^*, j) = a_{1j}p_1 + a_{2j}p_2 + \cdots + a_{mj}p_m \geq \nu^* =: v_1$$

such that $p_1 + p_2 + \cdots + p_m = 1$. Let us make change of variables as $\frac{p_i}{v_1} = x_i$, then we get

$$a_{1j}x_1 + a_{2j}x_2 + \cdots + a_{mj}x_m \geq 1$$

such that $x_1 + x_2 + \cdots + x_m = \frac{1}{v_1}$.

Note that player $A$ wants to maximize the reward $v_1$, i.e. $\frac{1}{v_1} \to \min$. This becomes the mathematical program: choose $(x_1, x_2, \ldots, x_m)$ to minimize objective function

$$z = \frac{1}{v_1} = x_1 + x_2 + \cdots + x_m \to \min \tag{1}$$

subject to constraints

$$\begin{aligned} a_{11}x_1 + a_{21}x_2 + \cdots + a_{m1}x_m &\geq 1 \\ a_{12}x_1 + a_{22}x_2 + \cdots + a_{m2}x_m &\geq 1 \\ &\vdots \\ a_{1n}x_1 + a_{2n}x_2 + \cdots + a_{mn}x_m &\geq 1 \\ x_1, x_2, \ldots, x_m &\geq 0 \end{aligned} \tag{2}$$

In a similar way, one may view the problem from player $B$'s point of view and arrive at a similar linear program. Problem of player $B$ is: choose $(y_1, y_2, \ldots, y_n)$ to maximize objective function

$$w = \frac{1}{v_2} = y_1 + y_2 + \cdots + y_n \to \max \tag{3}$$

subject to constraints

$$\begin{aligned}
a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n &\le 1 \\
a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n &\le 1 \\
&\vdots \\
a_{m1}y_1 + a_{m2}y_2 + \cdots + a_{mn}y_n &\le 1 \\
y_1, y_2, \ldots, y_n &\ge 0
\end{aligned} \tag{4}$$

In linear programming, there is a theory of duality (Lecture 5) that says these two programs, (1)-(2) and (3)-(4), are dual programs and can be solved by dual simplex method and simplex method, respectively. There is a remarkable theorem, called the Duality Theorem, that says dual programs have the same value. The maximum player $A$ can achieve in (3) is equal to the minimum that player $B$ can achieve in (1). In other words, $z_{\min} = w_{\max}$ or $\frac{1}{v_1} = \frac{1}{v_2} \Rightarrow v_1 = v_2 = \nu^*$. Let $(x_1^*, x_2^*, \ldots, x_m^*)$ and $(y_1^*, y_2^*, \ldots, y_n^*)$ be optimal solutions of (1)-(2) and (3)-(4), respectively. Then, the optimal mixed strategies $P^*$ and $Q^*$ can be found as $p_i^* = \nu^* x_i^*$ for $i = 1, 2, \ldots, m$ and $q_j^* = \nu^* y_j^*$ for $j = 1, 2, \ldots, n$, respectively.

**Example 9:** Recall rock-scissors-paper game considered in Example 3. Game matrix is

$$\mathcal{A} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

Now, we add $c = 1$ to each element of matrix $\mathcal{A}$. Then, we get the following table

| $A$ | $B$ | | | $\alpha_i$ |
|---|---|---|---|---|
| | $B_1$ | $B_2$ | $B_3$ | |
| $A_1$ | 1 | 2 | 0 | 0 |
| $A_2$ | 0 | 1 | 2 | 0 |
| $A_3$ | 2 | 0 | 1 | 0 |
| $\beta_j$ | 2 | 2 | 2 | $\alpha = 0$ $\beta = 2$ |

One can see that $\alpha \ne \beta$, and therefore the game is not solvable in pure strategies. Next, we can construct linear programming problems for players $A$ and $B$, respectively, as

$$\begin{aligned}
z = x_1 + x_2 + x_3 &\to \min \\
x_1 + 2x_3 &\ge 1 \\
2x_1 + x_2 &\ge 1 \\
2x_2 + x_3 &\ge 1 \\
x_1, x_2, x_3 &\ge 0
\end{aligned}
\qquad\qquad
\begin{aligned}
w = y_1 + y_2 + y_3 &\to \max \\
y_1 + 2y_2 &\le 1 \\
y_2 + 2y_3 &\le 1 \\
2y_1 + y_3 &\le 1 \\
y_1, y_2, y_3 &\ge 0
\end{aligned}$$

Solve this problem by simplex method. Then, the simplex tables for this system are

| $y_1$ | $y_2$ | $y_3$ | $s_1$ | $s_2$ | $s_3$ | $w$ | $b$ | variables | ratio | operations |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 0 | 1 | 0 | 0 | 0 | 1 | $s_1$ | 1 | $R_1 - \frac{1}{2}R_3$ |
| 0 | 1 | 2 | 0 | 1 | 0 | 0 | 1 | $s_2$ | | |
| **2** | 0 | 1 | 0 | 0 | 1 | 0 | 1 | $s_3$ | $\frac{1}{2}$ | $R_3 : 2$ |
| $-1$ | $-1$ | $-1$ | 0 | 0 | 0 | 1 | 0 | $w$ | | $R_4 + \frac{1}{2}R_3$ |
| 0 | **2** | $-\frac{1}{2}$ | 1 | 0 | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $s_1$ | $\frac{1}{4}$ | $R_1 : 2$ |
| 0 | 1 | 2 | 0 | 1 | 0 | 0 | 1 | $s_2$ | 1 | $R_2 - \frac{1}{2}R_1$ |
| 1 | 0 | $\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $y_1$ | | |
| 0 | $-1$ | $-\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | $w$ | | $R_4 + \frac{1}{2}R_2$ |
| 0 | 1 | $-\frac{1}{4}$ | $\frac{1}{2}$ | 0 | $-\frac{1}{4}$ | 0 | $\frac{1}{4}$ | $y_2$ | | $R_1 + \frac{1}{9}R_2$ |
| 0 | 0 | $\mathbf{\frac{9}{4}}$ | $-\frac{1}{2}$ | 1 | $\frac{1}{4}$ | 0 | $\frac{3}{4}$ | $s_2$ | $\frac{1}{3}$ | $\frac{4}{9} \cdot R_2$ |
| 1 | 0 | $\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $y_1$ | 1 | $R_3 - \frac{2}{9}R_2$ |
| 0 | 0 | $-\frac{3}{4}$ | $\frac{1}{2}$ | 0 | $\frac{1}{4}$ | 1 | $\frac{3}{4}$ | $w$ | | $R_4 - \frac{1}{3}R_2$ |

Finally, we get the following table

| $y_1$ | $y_2$ | $y_3$ | $s_1$ | $s_2$ | $s_3$ | $w$ | $b$ | variables |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | $\frac{1}{3}$ | $\frac{1}{3}$ | $-\frac{1}{6}$ | 0 | $\frac{1}{3}$ | $y_2$ |
| 0 | 0 | 1 | $-\frac{2}{3}$ | $-\frac{2}{3}$ | $\frac{1}{3}$ | 0 | $\frac{1}{3}$ | $y_3$ |
| 1 | 0 | 0 | $\frac{1}{3}$ | $-\frac{2}{3}$ | $\frac{1}{3}$ | 0 | $\frac{1}{3}$ | $y_1$ |
| 0 | 0 | 0 | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 1 | 1 | $w$ |

In the final table we see that there are no negative entries in the objective row. Hence, the optimal solution is found. Therefore, $w_{\max} = 1 = \nu^*$ and as a result the real value of the game is $\nu^* - c = 0$. Moreover, we can see that the optimal solutions for linear programs of player $A$ and $B$ are $y_1^* = y_2^* = y_3^* = \frac{1}{3}$ and $q_1^* = q_2^* = q_3^* = \frac{1}{3}$, respectively. Thus, the optimal mixed strategies are $P^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $Q^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

**Example 10:** Let the game matrix is

$$\mathcal{A} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 2 & -2 \end{pmatrix}$$

Formulate linear programming problem for the game and solve it.

Start from adding $c = 2$ to each element of matrix $\mathcal{A}$. Then, we get the following matrix

$$\tilde{\mathcal{A}} = \begin{pmatrix} 3 & 1 \\ 2 & 3 \\ 4 & 0 \end{pmatrix}$$

Now, we can formulate the linear program for player $A$:

$$\begin{aligned} 3p_1 + 2p_2 + 4p_3 &\geq \nu \\ p_1 + 3p_2 &\geq \nu \\ p_1 + p_2 + p_3 &= 1 \\ p_1, p_2, p_3 &\geq 0 \end{aligned}$$

Make change of variables $x_i = \frac{p_i}{\nu}$, for $i = 1, 2, 3$,

$$z = \frac{1}{\nu} = x_1 + x_2 + x_3 \to \min$$

subject to constraints

$$\begin{aligned} 3x_1 + 2x_2 + 4x_3 &\geq 1 \\ x_1 + 3x_2 &\geq 1 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

In a similar way, we can formulate the linear program for player $B$:

$$\begin{aligned} 3q_1 + q_2 &\leq \nu \\ 2q_1 + 3q_2 &\leq \nu \\ 4q_1 &\leq \nu \\ q_1 + q_2 &= 1 \\ q_1, q_2 &\geq 0 \end{aligned}$$

Make change of variables $y_j = \frac{q_j}{\nu}$, for $j = 1, 2$,

$$w = \frac{1}{\nu} = y_1 + y_2 \to \max$$

subject to constraints

$$\begin{aligned} 3y_1 + y_2 &\leq 1 \\ 2y_1 + 3y_2 &\leq 1 \\ 4y_1 &\leq 1 \\ y_1, y_2 &\geq 0 \end{aligned}$$

Solve this problem by simplex method. Then, the simplex tables for this system are

| $y_1$ | $y_2$ | $s_1$ | $s_2$ | $s_3$ | $w$ | $b$ | variables | ratio | operations |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 1 | 1 | 0 | 0 | 0 | 1 | $s_1$ | 1 | $R_1 - \frac{1}{3}R_2$ |
| 2 | **3** | 0 | 1 | 0 | 0 | 1 | $s_2$ | $\frac{1}{3}$ | $R_2 : 2$ |
| 4 | 0 | 0 | 0 | 1 | 0 | 1 | $s_3$ | | |
| $-1$ | $-1$ | 0 | 0 | 0 | 1 | 0 | $w$ | | $R_4 + \frac{1}{3}R_2$ |
| $\frac{7}{3}$ | 0 | 1 | $-\frac{1}{3}$ | 0 | 0 | $\frac{2}{3}$ | $s_1$ | $\frac{2}{7}$ | $R_1 - \frac{7}{12}R_3$ |
| $\frac{2}{3}$ | 1 | 0 | $\frac{1}{3}$ | 0 | 0 | $\frac{1}{3}$ | $y_2$ | $\frac{1}{2}$ | $R_2 - \frac{1}{6}R_3$ |
| **4** | 0 | 0 | 0 | 1 | 0 | 1 | $s_3$ | $\frac{1}{4}$ | $R_3 : 4$ |
| $-\frac{1}{3}$ | 0 | 0 | $\frac{1}{3}$ | 0 | 1 | $\frac{1}{3}$ | $w$ | | $R_4 + \frac{1}{12}R_3$ |

Finally, we get the following table

| $y_1$ | $y_2$ | $s_1$ | $s_2$ | $s_3$ | $w$ | $b$ | variables |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | $-\frac{1}{3}$ | $-\frac{7}{12}$ | 0 | $\frac{1}{12}$ | $s_1$ |
| 0 | 1 | 0 | $\frac{1}{3}$ | $-\frac{1}{6}$ | 0 | $\frac{1}{6}$ | $y_2$ |
| 1 | 0 | 0 | 0 | $\frac{1}{4}$ | 0 | $\frac{1}{4}$ | $y_1$ |
| 0 | 0 | 0 | $\frac{1}{3}$ | $\frac{1}{12}$ | 1 | $\frac{5}{12}$ | $w$ |

In the final table we see that there are no negative entries in the objective row. Hence, the optimal solution is found. Therefore, $w_{\max} = \frac{1}{\nu^*} = \frac{5}{12}$ or $\nu^* = \frac{12}{5}$ and as a result the real value of the game is $\nu^* - c = \frac{2}{5}$. Moreover, we can see that the optimal solution for linear program of player $B$ is $y_1^* = \frac{1}{4}$, $y_2^* = \frac{1}{6}$. Thus, the optimal mixed strategies can be calculated by $q_j^* = \nu^* y_j^*$ as $q_1^* = \frac{12}{5} \cdot \frac{1}{4} = \frac{3}{5}$ and $q_2^* = \frac{12}{5} \cdot \frac{1}{6} = \frac{2}{5}$ resulting in $Q^* = (\frac{3}{5}, \frac{2}{5})$. Analogously we can calculate the optimal mixed strategies for player $A$. Using the fact that both problems are dual we can find the optimal solution for the linear program of player $A$ as $x_1^* = 0$, $x_2^* = \frac{1}{3}$, $x_3^* = \frac{1}{12}$. Thus, using $p_i^* = \nu^* x_i^*$, we get $p_1^* = \frac{12}{5} \cdot 0 = 0$, $p_2^* = \frac{12}{5} \cdot \frac{1}{3} = \frac{4}{5}$, and $p_3^* = \frac{12}{5} \cdot \frac{1}{12} = \frac{1}{5}$ resulting in $P^* = (0, \frac{4}{5}, \frac{1}{5})$.

## Exercises

**Example 11:** Two stores, $A$ and $B$, are planning to locate in one of two towns. Town I has 60 percent of population while town II has 40 percent. If both stores locate in the same town they will split the total business of both towns equally, but if they locate in different towns each will get the business of that town. Where should each store locate?

**Example 12:** Peter and Paul decide to play the following game: each one of them simultaneously display 1, 2, or 5 cents. If both show the same number of pennies, no money is exchanged; but if they show different numbers, Peter gets odd sums and Paul gets even sums. Set up the payoff matrix, find the value of the game and the optimal strategies of the players.

**Example 13:** Solve the games with the following matrices:

$$\mathcal{A}_1 = \begin{pmatrix} 1 & 2 & 0 \\ 2 & -1 & 3 \end{pmatrix}, \quad \mathcal{A}_2 = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 2 & 1 \\ 3 & 1 & 2 & 2 \end{pmatrix}, \quad \mathcal{A}_3 = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 2 & 1 \\ 3 & 1 & 2 & 2 \end{pmatrix},$$

$$\mathcal{A}_4 = \begin{pmatrix} 4 & 1 & -3 \\ 3 & 2 & 5 \\ 0 & 1 & 6 \end{pmatrix}, \quad \mathcal{A}_5 = \begin{pmatrix} 0 & 5 & -2 \\ -3 & 0 & 4 \\ 6 & -4 & 0 \end{pmatrix}, \quad \mathcal{A}_6 = \begin{pmatrix} 1 & 0 & 3 \\ -2 & 3 & 0 \\ -4 & 5 & -6 \end{pmatrix}.$$

## Problems

**8.1:** Solve the games with the following matrices:

$$\mathcal{A}_1 = \begin{pmatrix} 2 & 5 & 3 \\ 6 & 4 & 5 \\ 3 & 7 & 6 \\ 2 & 6 & 4 \end{pmatrix}, \qquad \mathcal{A}_2 = \begin{pmatrix} 0 & 3 & -3 \\ 3 & 9 & -6 \\ 3 & -1 & 2 \end{pmatrix}, \qquad \mathcal{A}_3 = \begin{pmatrix} 3 & 2 & -4 & 6 \\ -2 & 5 & -1 & 2 \end{pmatrix},$$

$$\mathcal{A}_4 = \begin{pmatrix} 6 & 2 & 5 \\ 4 & 3 & 7 \\ 5 & 5 & 6 \end{pmatrix}, \qquad \mathcal{A}_5 = \begin{pmatrix} 4 & 0 & 5 & -2 \\ 2 & 6 & 1 & 7 \end{pmatrix}.$$

**8.2:** Player $A$ holds a black Ace and a red 8. Player $B$ holds a red 2 and a black 7. The players simultaneously choose a card to play. If the chosen cards are of the same color, player $A$ wins. Player $B$ wins if the cards are of different colors. The amount won is a number of dollars equal to the number on the winner's card (Ace counts as 1.) Set up the payoff function, find the value of the game and the optimal strategies of the players.

## Answers to problems

1. $\mathcal{A}_1$: $\nu^* = 5$, $P^* = \left(0, \frac{2}{3}, \frac{1}{3}, 0\right)$, $Q^* = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$;
   $\mathcal{A}_2$: $\nu^* = \frac{2}{3}$, $P^* = \left(0, \frac{1}{6}, \frac{5}{6}\right)$, $Q^* = \left(0, \frac{4}{9}, \frac{5}{9}\right)$;
   $\mathcal{A}_3$: $\nu^* = -\frac{11}{8}$, $P^* = \left(\frac{1}{8}, \frac{7}{8}\right)$, $Q^* = \left(\frac{3}{8}, 0, \frac{5}{8}, 0\right)$;
   $\mathcal{A}_4$: $\nu^* = 5$, $P^* = (0, 0, 1)$, $Q^* = (0, 1, 0)$;
   $\mathcal{A}_5$: $\nu^* = \frac{32}{11}$, $P^* = \left(\frac{5}{11}, \frac{6}{11}\right)$, $Q^* = \left(\frac{9}{11}, 0, 0, \frac{2}{11}\right)$.

2. The payoff tableau is

| | red 2 | black 7 |
|---|---|---|
| black Ace | $-2$ | 1 |
| red 8 | 8 | $-7$ |

   The value of the game is $\nu^* = -\frac{1}{3}$, and the optimal mixed strategies for players $A$ and $B$ are $P^* = (\frac{5}{6}, \frac{1}{6})$ and $Q^* = (\frac{5}{9}, \frac{4}{9})$, respectively.