ISS0031 Modeling and Identification

Lecture 5

Introduction

In the previous lesson we have seen that in the case of a maximum problem the Simplex method can be applied easily in order to find the optimal solution. Whereas when the problem is of minimization we have to take recourse to the Big M method or alternative methods, which are, in general, a bit complicated. Among a number of techniques for solving such problems is one developed by John Von Neumann and others, in which the solution (if it exists) of a minimum problem is found by solving a related maximum problem called the dual problem.

The concept of duality plays a very important role in the development of linear programming theory. The concept of a dual problem is sometimes very fruitful when the dual is easier to solve than that of the primal. A particular case when the number of variables is less than the constraints in a linear programming problem, its solution can be found more easily by solving its dual instead the primal problem.

Primal and dual linear programming problem

Let linear programming problem is given in the standard form

$$z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \to \max \tag{1}$$

subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\le b_1 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\le b_m \end{aligned} \tag{2}$$

and

$$x_1 \ge 0, x_2 \ge 0, \ldots, x_n \ge 0 \tag{3}$$

The linear programming problem (1)-(3) is called the **primal problem**. Recall that it can be written in the matrix notation as

$$\begin{aligned} z = CX &\to \max \\ AX &\le B \\ X &\ge 0 \end{aligned}$$

where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad C = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}.$$

Definition 1. *The **dual** of the standard maximum problem (1)-(3) is defined to be the standard minimum problem*

$$w = b_1y_1 + b_2y_2 + \cdots + b_my_m \to \min$$

subject to the constraints

$$\begin{aligned} a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m &\geq c_1 \\ &\vdots \\ a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m &\geq c_n \end{aligned}$$

and

$$y_1 \geq 0, y_2 \geq 0, \ldots, y_n \geq 0.$$

Definition 2. *The problem (1)-(3) is called symmetrical dual problem.*

Similarly, the dual linear programming problem can be written in the matrix form as

$$\begin{aligned} w = B^TY &\to \min \\ A^TY &\geq C^T \\ Y &\geq 0 \end{aligned}$$

where $Y = \begin{bmatrix} y_1 & y_2 & \ldots & y_m \end{bmatrix}^T$.

Example 1: Obtain the dual of the minimum problem

$$z = 50x_1 + 25x_2 \to \min$$

subject to the constraints

$$\begin{aligned} x_1 + 3x_2 &\geq 8 \\ 3x_1 + 4x_2 &\geq 19 \\ 3x_1 + x_2 &\geq 7 \\ x_1 \geq 0, x_2 &\geq 0 \end{aligned}$$

Here the minimum problem is in the standard form. We begin by constructing a special matrix for the coefficients of the constraints of this problem without introducing slack variables. As in a simplex table, we place the objective function in the last row. The special table is then

x_1	x_2	b
1	3	8
3	4	19
3	1	7
50	25	0

Taking transpose of the matrix associated to this table, i.e. interchanging the rows to columns, we get

y_1	y_2	y_3	c
1	3	3	50
3	4	1	25
8	19	7	0

Translating this table into a maximum problem in standard form, we get

$$w = 8y_1 + 19y_2 + 7y_3 \to \max$$

subject to the constraints

$$\begin{aligned} y_1 + 3y_2 + 3y_3 &\leq 50 \\ 3y_1 + 4y_2 + y_3 &\leq 25 \\ y_1 \geq 0, y_2 \geq 0, y_3 &\geq 0 \end{aligned}$$

This is the dual of the given minimum problem.

Table 1: Relations between primal and dual problems

Primal	Dual
constraints	variables
variables	constraints
max	min
min	max
all constraints are $\geq$ or $\leq$	all constraints are $\leq$ or $\geq$
ith constraint contains $=$	ith variable has undefined sign
jth variable has undefined sign	jth constraint contains $=$
A is a matrix of constraints	A^T is a matrix of constraints
free terms (b_j)	coefficients of objective function (c_i)
coefficients of objective function (c_i)	free terms (b_j)

Example 2: Obtain the dual of the following linear programming problem

$$z = x_1 - 4x_2 - 3x_3 \to \max$$

subject to the constraints

$$\begin{aligned} 3x_1 + 4x_2 + x_3 &\geq 7 \\ x_1 + 2x_2 + x_3 &= 6 \\ x_3 &\leq 4 \\ x_1 \geq 0, x_2 \geq 0, x_3 &\geq 0 \end{aligned}$$

The set of constraints can be rewritten as

$$\begin{aligned} 3x_1 + 4x_2 + x_3 &\geq 7 \\ -x_1 - 2x_2 - x_3 &\leq -6 \\ x_1 + 2x_2 + x_3 &\leq 6 \\ x_3 &\leq 4 \\ x_1 \geq 0, x_2 \geq 0, x_3 &\geq 0 \end{aligned}$$

Then, the dual problem is

$$w = -7y_1 - 6y_2' + 6y_2'' + 4y_3 \to \min$$

subject to the constraints

$$\begin{aligned} -3y_1 - y_2' + y_2'' &\geq 1 \\ -4y_1 - 2y_2' + 2y_2'' &\geq -4 \\ -y_1 - y_2' + y_2'' + y_3 &\geq -3 \\ y_1 \geq 0, y_2' \geq 0, y_2'' \geq 0, y_3 &\geq 0 \end{aligned}$$

or, using substitution $y_2' - y_2'' = y_2$, we get

$$\begin{aligned} w = -7y_1 - 6y_2 + 4y_3 &\to \min \\ -3y_1 - y_2 &\geq 1 \\ -4y_1 - 2y_2 &\geq -4 \\ -y_1 - y_2 + y_3 &\geq -3 \\ y_1 \geq 0, y_3 &\geq 0 \\ y_2 &\text{ free} \end{aligned}$$

Example 3: Obtain the dual of the following linear programming problem

$$z = 6x_1 + 10x_2 \to \min$$

subject to the constraints

$$\begin{aligned} 5x_1 + 3x_2 &\geq 10 \\ x_1 - x_2 &\leq 4 \\ x_1 &\geq 0 \\ x_2 &\text{ free} \end{aligned}$$

Let $x_2 = x_2' - x_2''$ such that $x_2', x_2'' \geq 0$, then

$$z = 6x_1 + 10x_2' - 10x_2'' \to \min$$

subject to the constraints

$$\begin{aligned} 5x_1 + 3x_2' - 3x_2'' &\geq 10 \\ x_1 - x_2' + x_2'' &\leq 4 \\ x_1 \geq 0, x_2' \geq 0, x_2'' &\geq 0 \end{aligned} \qquad \text{or} \qquad \begin{aligned} 5x_1 + 3x_2' - 3x_2'' &\geq 10 \\ -x_1 + x_2' - x_2'' &\geq -4 \\ x_1 \geq 0, x_2' \geq 0, x_2'' &\geq 0 \end{aligned}$$

The dual problem is

$$w = 10y_1 - 4y_2 \to \max$$

subject to the constraints

$$\begin{aligned} 5y_1 - y_2 &\le 6 \\ 3y_1 + y_2 &\le 10 \\ -3y_1 - y_2 &\le -10 \\ y_1 \ge 0, y_2 &\ge 0 \end{aligned} \qquad \text{or} \qquad \begin{aligned} 5y_1 - y_2 &\le 6 \\ 3y_1 + y_2 &= 10 \\ y_1 \ge 0, y_2 &\ge 0 \end{aligned}$$

Duality theorems

Recall that

$$\begin{aligned} z = CX &\to \max \\ AX &\le B \\ X &\ge 0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} w = B^TY &\to \min \\ A^TY &\ge C^T \\ Y &\ge 0 \end{aligned}$$

are symmetrical dual problems.

Theorem 1. *If X_0 is feasible solution for the standard maximum problem and Y_0 is feasible for for its dual, then $CX_0 \le B^TY_0$.*

Proof: Since both X_0 and Y_0 are feasible solutions, $AX_0 \le B$ and $A^TY_0 \ge C^T$. Multiplying the inequality $AX_0 \le B$ by Y_0^T from the left, we get $Y_0^TAX_0 \le Y_0^TB$. Note that $CX_0 = (C^T)^TX_0 \le (A^TY_0)^TX_0 = Y_0^TAX_0 \le Y_0^TB = B^TY_0$ and as a result $CX_0 \le B^TY_0$. ■

Corollary 1. *If a primal and dual problems are both feasible, then both are bounded feasible.*

Theorem 2. *If X^* and Y^* are feasible solutions for the symmetrical dual problems and $CX^* = B^TY^*$, then X^* and Y^* are optimal solutions.*

Proof: Suppose that X is a feasible solution for the primal problem, then, by Theorem 1, we get $CX \le B^TY^* = CX^*$ or $CX \le CX^*$, which shows (Definition 5, Lecture 1) that X^* is the optimal solution. A symmetric argument works for Y^*. ■

Theorem 3 (**The duality theorem**). *If a standard linear programming problem is bounded feasible, then so is its dual, their values are equal, and there exist optimal vectors for both problems.*

Theorem 4 (**The equilibrium theorem**). *Let X^* and Y^* be feasible solutions for the symmetrical dual problems, then X^* and Y^* are optimal if and only if*

$$y_i^* = 0 \quad \textit{for all} \quad i \quad \textit{for which} \quad \sum_{j=1}^{n} a_{ij}x_j^* < b_i \tag{4}$$

and

$$x_j^* = 0 \quad \textit{for all} \quad j \quad \textit{for which} \quad \sum_{i=1}^{m} y_i^*a_{ij} > c_j. \tag{5}$$

There are three possibilities for a linear programm. It may be feasible bounded (f.b.), feasible unbounded (f.u.), or infeasible (i). For a programm and its dual, there are therefore nine possibilities. Corollary 1 states that three of these cannot occur: If a problem and its dual are both feasible, then both must be bounded feasible. The first conclusion of Theorem 3 states that two other possibilities cannot occur. If a program is feasible bounded, its dual cannot be infeasible. The x's in the accompanying diagram show the impossibilities. The remaining four possibilities can occur.

Primal problem

		f.b.	f.u.	i.
	f.b.		x	x
Dual	f.u.	x	x	
	i.	x		

Dual Simplex method

The detailed steps of dual Simplex method:

Step 1: Start with a dual feasible basis. Create the initial simplex tableau by creating an augmented matrix from the equations, placing the equation for the objective function last. If the right-hand side of each constraint is nonnegative, the optimal solution has been found; if not, at least one constraint has a negative right-hand side, and go to Step 2.

Step 2: Choose the most negative basic variable as the variable to leave the basis. The row in which the variable is basic will be the pivot row. To select the variable that enters the basis, we compute the following ratio for each variable x_j that has a negative coefficient in the pivot row: (coefficient of x_j in the objective row)/(coefficient of x_j in pivot row). Choose the variable with the smallest ratio (absolute value) as the entering variable. This form of the ratio test maintains a dual feasible tableau (all variables in the objective row has non-negative coefficients).

Step 3: If there is any constraint in which the right-hand side is negative and each variable has a nonnegative coefficient, the linear programming problem has no feasible solution. If no constraint indicating feasibility is found, return to Step 2.

Step 4: When the final matrix has been obtained, determine the final basic solution. This will give the maximum value for the objective function and the values of the variables where this maximum occurs.

Example 4: Solve the following linear programming problem.

$$z = -5x_1 - 35x_2 - 20x_3 \to \max$$
$$x_1 - x_2 - x_3 \leq -2$$
$$-x_1 - 3x_2 \leq -3$$
$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

This particular example is a maximum problem containing three variables x_1, x_2, x_3 and it is written in standard form. In this problem we have the constraints as linear expressions less than or equal to some constants. That means there is a slack between the left and right sides of the inequalities. In order to take up the slack between the left and right sides of the problem, let us introduce the slack variables s_1 and s_2 which are greater than or equal to zero, such that

$$x_1 - x_2 - x_3 + s_1 = -2$$
$$-x_1 - 3x_2 + s_2 = -3$$

Furthermore, the objective function z can be rewritten as $z + 5x_1 + 35x_2 + 20x_3 = 0$. As a result, we have now replaced our original system of constraints and the objective function by a system of three equations in six unknowns as

$$z + 5x_1 + 35x_2 + 20x_3 = 0$$
$$x_1 - x_2 - x_3 + s_1 = -2$$
$$-x_1 - 3x_2 + s_2 = -3$$

Here, we have to find the particular solution $(x_1, x_2, x_3, s_1, s_2, z)$ that gives the largest possible value for z. Then we can construct the initial simplex table for this system which is primal infeasible but dual feasible. Therefore, we solve it using dual Simplex method.

x_1	x_2	x_3	s_1	s_2	z	b	variables
1	−1	−1	1	0	0	−2	s_1
−1	−3	0	0	1	0	−3	s_2
5	35	20	0	0	1	0	z

From this point on, the simplex method consists of pivoting from one table to another until the optimal solution is found.

Pivot element: The pivot element for the dual Simplex method is found using the following rules:

- The pivot row is selected by locating the most negative entry in the column b of the right-hand sides of equations. If there are several the most negative entries, then choose any of them.
- To select the pivot column, we compute the following ratio for each variable x_j that has a negative coefficient in the pivot row: (coefficient of x_j in the objective row)/(coefficient of x_j in pivot row). Choose the column in which the smallest positive ratio (absolute value) is obtained as the pivot column. If all the variables in the pivot row are positive, then the problem is infeasible.

The pivot element is the entry at the intersection of the pivot row and the pivot column.

In this example, -3 is the most negative entry in the last column, so the second row is the pivot row. On dividing all the entries in the objective row by the corresponding entry in the second row, we get 5 as the smallest ratio, so the first column is the pivot column. Hence, $a_{21} = -1$ is the pivot element, which is marked in the initial table by a bold font.

x_1	x_2	x_3	s_1	s_2	z	b	variables
1	-1	-1	1	0	0	-2	s_1
$\mathbf{-1}$	-3	0	0	1	0	-3	s_2
5	35	20	0	0	1	0	z
5	11.67						ratio

Next, we divide R_2 (the second row) by -1 and then apply the operations $R_1 - R_2$ and $R_3 - 5R_2$. The feasibility test at Step 3 fails because the basis is not yet primal feasible so we return to Step 2.

x_1	x_2	x_3	s_1	s_2	z	b	variables
0	$\mathbf{-4}$	-1	1	1	0	-5	s_1
1	3	0	0	-1	0	3	x_1
0	20	20	0	5	1	-15	z
	5	20					ratio

The first row is selected as the pivot row, so s_1 leaves the basis and x_2 enters. Now, we divide R_1 by -4 and then apply the operations $R_2 - 3R_1$ and $R_3 - 20R_1$. This leads to the tableau below that still has a negative entries in the last column, so we return to Step 2.

x_1	x_2	x_3	s_1	s_2	z	b	variables
0	1	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	0	$\frac{5}{4}$	x_2
1	0	$\mathbf{-\frac{3}{4}}$	$\frac{3}{4}$	$-\frac{1}{4}$	0	$-\frac{3}{4}$	x_1
0	0	15	5	10	1	-40	z
		20		40			ratio

Selecting the second row as the pivot row, x_1 leaves the basis and x_3 enters. Next, we divide R_2 by $-\frac{4}{3}$ and apply the operations $R_1 - \frac{1}{4}R_2$ and $R_3 - 15R_2$.

x_1	x_2	x_3	s_1	s_2	z	b	variables
$\frac{1}{3}$	1	0	0	$-\frac{1}{3}$	0	1	x_2
$-\frac{4}{3}$	0	1	-1	$\frac{1}{3}$	0	1	x_3
20	0	0	20	5	1	-55	z

The updated tableau is both primal and dual feasible indicating that the optimal solution has been obtained. Therefore, $z_{\max}(0, 1, 1) = -55$.

Exercises

Solve the following linear programming problem by dual Simplex method.

Example 5:

$$\begin{aligned} z = -x_1 - 2x_2 - 3x_3 \rightarrow \max \\ x_1 - 2x_2 + 3x_3 \geq -1 \\ 2x_1 - x_2 - x_3 \leq -1 \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{aligned}$$

Obtain the dual of the following linear programming problems and solve them.

Example 6:

$$\begin{aligned} z = 2x_1 + x_3 \rightarrow \min \\ x_1 + x_2 - x_3 \geq 5 \\ x_1 - 2x_2 + 4x_3 \geq 8 \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{aligned}$$

Example 7:

$$\begin{aligned} z = 50x_1 + 45x_2 + 40x_3 \rightarrow \max \\ 50x_1 + 25x_2 + 25x_3 \leq 1450 \\ 35x_1 + 10x_2 + 20x_3 \leq 700 \\ 35x_1 + 15x_2 + 15x_3 \leq 1050 \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{aligned}$$

Problems

Solve the following linear programming problems using Dual Simplex Method.

5.1:

$$\begin{aligned} z = 5x_1 + 12x_2 + 8x_3 + 11x_4 \rightarrow \min \\ x_1 + 2x_2 + x_3 + 2x_4 \geq 12 \\ x_1 + 4x_2 - 3x_3 + 8x_4 \geq 6 \\ -x_1 - 5x_2 + x_3 - x_4 \geq -7 \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0 \end{aligned}$$

Obtain the dual of the following linear programming problems and solve them.

5.2:

$$
\begin{aligned}
z = 40x_1 + 36x_2 + 32x_3 &\to \max \\
40x_1 + 20x_2 + 20x_3 &\leq 1160 \\
28x_1 + 8x_2 + 16x_3 &\leq 560 \\
28x_1 + 12x_2 + 12x_3 &\leq 840 \\
x_1 \geq 0, x_2 \geq 0, x_3 &\geq 0
\end{aligned}
$$

5.3:

$$
\begin{aligned}
z = 70x_1 + 70x_2 + 60x_3 &\to \max \\
30x_1 + 30x_2 + 90x_3 &\leq 8100 \\
100x_1 + 90x_2 + 150x_3 &\leq 9000 \\
50x_1 + 50x_2 + 10x_3 &\leq 2500 \\
x_1 \geq 0, x_2 \geq 0, x_3 &\geq 0
\end{aligned}
$$

5.4:

$$
\begin{aligned}
z = 60x_1 + 54x_2 + 48x_3 &\to \max \\
60x_1 + 30x_2 + 30x_3 &\leq 1740 \\
42x_1 + 12x_2 + 24x_3 &\leq 840 \\
42x_1 + 18x_2 + 18x_3 &\leq 1260 \\
x_1 \geq 0, x_2 \geq 0, x_3 &\geq 0
\end{aligned}
$$

5.5:

$$
\begin{aligned}
z = 9x_1 + 14x_2 + x_3 &\to \max \\
9x_1 + 4x_2 + 4x_3 &\leq 54 \\
9x_1 + 5x_2 + 5x_3 &\leq 63 \\
x_2 &\leq 5 \\
x_1 \geq 0, x_2 \geq 0, x_3 &\geq 0
\end{aligned}
$$

Answers to problems

1. $z_{\min}(2,0,0,5,0,36,0) = 65$.

2. The dual problem is

$$\begin{aligned} w = 1160y_1 + 560y_2 + 840y_3 &\to \min \\ 40y_1 + 28y_2 + 28y_3 &\geq 40 \\ 20y_1 + 8y_2 + 12y_3 &\geq 36 \\ 20y_1 + 16y_2 + 12y_3 &\geq 32 \\ y_1 \geq 0, y_2 \geq 0, y_3 &\geq 0 \end{aligned}$$

   and $w_{\min}\left(\frac{9}{5},0,0,32,0,4\right) = 2088$.

3. The dual problem is

$$\begin{aligned} w = 8100y_1 + 9000y_2 + 250y_3 &\to \min \\ 30y_1 + 100y_2 + 50y_3 &\geq 70 \\ 30y_1 + 90y_2 + 50y_3 &\geq 70 \\ 90y_1 + 150y_2 + 10y_3 &\geq 60 \\ y_1 \geq 0, y_2 \geq 0, y_3 &\geq 0 \end{aligned}$$

   and $w_{\min}\left(0,\frac{23}{66},\frac{17}{22},\frac{115}{33},0,0\right) = \frac{55750}{11}$.

4. The dual problem is

$$\begin{aligned} w = 1740y_1 + 840y_2 + 1260y_3 &\to \min \\ 60y_1 + 42y_2 + 42y_3 &\geq 60 \\ 30y_1 + 12y_2 + 18y_3 &\geq 54 \\ 30y_1 + 24y_2 + 18y_3 &\geq 48 \\ y_1 \geq 0, y_2 \geq 0, y_3 &\geq 0 \end{aligned}$$

   and $w_{\min}\left(\frac{9}{5},0,0,48,0,6\right) = 3132$.

5. The dual problem is

$$\begin{aligned} w = 54y_1 + 63y_2 + 5y_3 &\to \min \\ 9y_1 + 9y_2 &\geq 9 \\ 4y_1 + 5y_2 + y_3 &\geq 14 \\ 4y_1 + 5y_2 &\geq 5 \\ y_1 \geq 0, y_2 \geq 0, y_3 &\geq 0 \end{aligned}$$

   and $w_{\min}(0,1,9,0,0,0) = 108$.