ISS0031 Modeling and Identification

Lecture 4

Introduction

The graphical method for solving linear programming problems is useful when there are exactly two variables and the number of constraints is relatively small. Note that, if we have a large number of either variables or constraints, it is still true that the optimal solution will be found at the vertex of the set of feasible solutions (Lecture 3, Theorem 2). In fact, we could find these vertices by writing all the equations corresponding to the inequalities of the problem and then proceeding to solve all possible combinations of these equations. After that we would have to discard any solutions that are not feasible. Then we could evaluate the objective function at the remaining feasible solutions. After all this, we might discover that the problem has no optimal solution at all. However, the described procedure seems to be very complex. For example, if there were just 3 variables and 5 constraints, we would have to solve all possible combinations of 3 equations chosen from a set of 5 equations. It is not hard to see that there will be $\binom{3}{5} = 10$ solutions. Each of these solutions will then have to be tested for feasibility. So even for this relatively small number of variables and constraints, the work would be quite tedious.

However, in the real world of applications there are usually hundreds and even thousands of variables involved. Therefore, we need a more systematic approach which allows to simplify the searching procedure. One very efficient technique of doing this is a **Simplex method**.

Simplex method

Let us start from the illustrative problem in which the objective function is to be maximized.

Example 1: Solve the following linear programming problem.

$$\begin{aligned} z = 3x_1 + 4x_2 &\to \max \\ 2x_1 + 4x_2 &\leq 120 \\ 2x_1 + 2x_2 &\leq 80 \\ x_1 \geq 0, x_2 &\geq 0 \end{aligned}$$

The procedure for solving the given problem is illustrated in the following steps:

Step 1: Standard form of a maximum problem

A linear programming problem in which the objective function is to be maximized is referred to as a maximum linear programming problem. Such problems are said to be in standard form if the following conditions are satisfied:

- all the variables are nonnegative;

- all the other constraints are written as a linear expression, that is, less than or equal to a positive constant.

This particular example is a maximum problem containing two variables x_1 and x_2. Since both the variables are nonnegative and the other constraints are each written as linear expression less than or equal to a positive constant, therefore, we conclude that the maximum problem is in standard form.

Steep 2: Slack variables and initial simplex table

In order to solve the maximum problem by simplex method, we need to do the following first:

- introduction of slack variables;
- construction of the initial simplex table.

In this problem we have the constraints as linear expressions less than or equal to some positive constants. That means there is a slack between the left and right sides of the inequalities. In order to take up the slack between the left and right sides of the problem, let us introduce the slack variables s_1 and s_2 which are greater than or equal to zero, such that

$$\begin{aligned} 2x_1 + 4x_2 + s_1 &= 120 \\ 2x_1 + 2x_2 + s_2 &= 80 \end{aligned}$$

Furthermore, the objective function z can be rewritten as $z - 3x_1 - 4x_2 = 0$. In effect, we have now replaced our original system of constraints and the objective function by a system of three equations in five unknowns as

$$\begin{aligned} z - 3x_1 - 4x_2 &= 0 \\ 2x_1 + 4x_2 + s_1 &= 120 \\ 2x_1 + 2x_2 + s_2 &= 80 \end{aligned}$$

Here, we have to find the particular solution (x_1, x_2, s_1, s_2, z) that gives the largest possible value for z. Then we can construct the initial (starting) simplex table (matrix) for this system as

x_1	x_2	s_1	s_2	z	b	variables
2	4	1	0	0	120	s_1
2	2	0	1	0	80	s_2
-3	-4	0	0	1	0	z

Notice that the coefficients of the objective function are arranged in the bottom row which is called the objective row.

From this point on, the simplex method consists of pivoting from one table to another until the optimal solution is found.

Pivoting: To pivot a matrix about a given element, called the pivot element, is to apply row operations so that the pivot element is replaced by 1 and all other entries in the same column (called pivot column) become 0. More specifically, in the pivot row, divide each entry by the pivot element (we assume it is not 0). Obtain 0 elsewhere in the pivot column by performing row operations.

Pivot element: The pivot element for the Simplex method is found using the following rules:

- The pivot column is selected by locating the most negative entry in the objective row. If all the entries in this column are negative, the problem is unbounded and there is no solution.
- Divide each entry in the last column by the corresponding entry (from the same row) in the pivot column. (Ignore the rows in which the pivot column entry is less than or equal to 0). The row in which the smallest positive ratio is obtained is the pivot row.

The pivot element is the entry at the intersection of the pivot row and the pivot column.

In this example, -4 is the most negative entry in the objective row, so the second column is the pivot column. On dividing all the entries in the fifth column by the corresponding entry in the second column, we get 30 as the smallest ratio, so the first row is the pivot row. Hence, $a_{21} = 4$ is the pivot element, which is marked in the initial table (reproduced below) by a bold font.

x_1	x_2	s_1	s_2	z	b	variables	ratio
2	**4**	1	0	0	120	s_1	$\frac{120}{4} = 30$
2	2	0	1	0	80	s_2	$\frac{80}{2} = 40$
-3	-4	0	0	1	0	z	

Next, we divide R_1 (the first row) by 4 and then apply the operations $R_2 - 2R_1$ and $R_3 + 4R_1$. The new table becomes

x_1	x_2	s_1	s_2	z	b	variables	ratio
$\frac{1}{2}$	1	$\frac{1}{4}$	0	0	30	x_2	60
1	0	$-\frac{1}{2}$	1	0	20	s_2	20
-1	0	1	0	1	120	z	

In the second simplex table -1 is the most negative entry in the objective row, so the first column is the pivot column. On dividing all the entries in the the fifth column by the corresponding entry in the first column, we get 20 as the smallest ratio, so the second row is the pivot row. Hence, $a_{21} = 1$ is the pivot element. This is marked by a bold font in the second simplex table. Next, apply the operations $R_1 - \frac{1}{2}R_2$ and $R_3 + R_2$. The new table becomes

x_1	x_2	s_1	s_2	z	b	variables
0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	20	x_2
1	0	$-\frac{1}{2}$	1	0	20	x_1
0	0	$\frac{1}{2}$	1	1	140	z

In the above table we notice that there are no negative entries in the objective row. Hence, the optimal solution has been found. Therefore, $z_{\max}(20, 20) = 140$.

Note that, in general, a minimum problem can be changed to a maximum problem by realizing that in order to minimize z we must maximize $-z$. That is in such cases we multiply the objective function by 1 and convert it into a maximum problem and solve it as discussed above.

Example 2: Consider the following linear programming problem.

$$
\begin{aligned}
z = x_1 - 3x_2 + 2x_3 &\to \min \\
3x_1 - x_2 + 2x_3 &\leq 7 \\
-2x_1 + 4x_2 &\leq 12 \\
-4x_1 + 3x_2 + 8x_3 &\leq 10 \\
x_1 \geq 0, x_2 \geq 0, x_3 &\geq 0
\end{aligned}
$$

This is a problem of minimization in which all the constraints are written as a linear expression, that is, less than or equal to a positive constant. Therefore, converting the objective function for maximization, we have $z' = -x_1 + 3x_2 - 2x_3 \to \max$, where $z' = -z$.

After introducing the slack variable the problem can be expressed as

$$
\begin{aligned}
3x_1 - x_2 + 2x_3 + s_1 &= 7 \\
-2x_1 + 4x_2 + s_2 &= 12 \\
-4x_1 + 3x_2 + 8x_3 + s_3 &= 10 \\
x_1 \geq 0, x_2 \geq 0, x_3 &\geq 0 \\
s_1 \geq 0, s_2 \geq 0, s_3 &\geq 0
\end{aligned}
$$

The objective function can be written as $z' + x_1 - 3x_2 + 2x_3 = 0$. The initial simplex table for this system is

x_1	x_2	x_3	s_1	s_2	s_3	z'	b	variables	ratio
3	−1	2	1	0	0	0	7	s_1	$\frac{12}{4} = 3$
−2	**4**	0	0	1	0	0	12	s_2	$\frac{10}{3}$
−4	3	8	0	0	1	0	10	s_3	
1	−3	2	0	0	0	1	0	z	

The pivot column is found by locating the column containing the smallest entry in the objective row (− 3 in the second column). The pivot row is obtained by dividing each entry in the fifth column by the corresponding entry in the pivot column (ignoring row in which the pivot column contains a negative number) and selecting the smallest non-negative ratio. Thus, the second row is the pivot row and the pivot element in that row is the bolded element 4. Diving R_2 by 4 and applying the operations R_1+R_2, R_3-3R_2, and R_4+3R_2, we get the second simplex table as

x_1	x_2	x_3	s_1	s_2	s_3	z	b	variables
$\mathbf{\frac{5}{2}}$	0	2	1	$\frac{1}{4}$	0	0	10	s_1
$-\frac{1}{2}$	1	0	0	$\frac{1}{4}$	0	0	3	x_2
$-\frac{5}{2}$	0	8	0	$-\frac{3}{4}$	1	0	1	s_3
$-\frac{1}{2}$	0	2	0	$\frac{3}{4}$	0	1	9	z

Applying the same procedure, we determine the next pivot element to be $\frac{5}{2}$ in the first column and first row. Dividing R_1 by $\frac{5}{2}$ and applying the operations $R_2+\frac{1}{2}R_1$, $R_3+\frac{5}{2}R_1$, and $R_4+\frac{1}{2}R_1$, we get the third simplex table as

x_1	x_2	x_3	s_1	s_2	s_3	z	b	variables
1	0	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{1}{10}$	0	0	4	x_1
0	1	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{3}{10}$	0	0	5	x_2
0	0	10	1	$-\frac{1}{2}$	1	0	11	s_3
0	0	$\frac{12}{5}$	$\frac{1}{5}$	$\frac{4}{5}$	0	1	11	z

In the above table we see that there are no negative entries in the objective row. Hence, the optimal solution is found. Therefore, $z_{\max}=11$ for $x_1=4, x_2=5, x_3=0, s_1=0, s_2=0, s_3=11$. Hence, the solution of the original problem is $z_{\min}(4,5,0)=-11$.

The big M method

Next, we present a brief description of the so-called big M method.

1. Modify the constraints so that the right-hand-side of each constraint is non-negative. Identify each constraint that is now an $=$ or $\geq$ constraint.

2. Convert each inequality constraint to canonical form (add a slack variable for $\leq$ constraints and a surplus variable for $\geq$ constraints).

3. For each $\geq$ or $=$ constraint, add artificial variables such that $y_i \geq 0$.

4. Let M denote a very large positive number. If the linear programming problem is a maximization problem, add (for each artificial variable) $-My_i$ to the objective function. Otherwise, add (for each artificial variable) My_i to the objective function.

5. Since each artificial variable will be in the starting basis, all artificial variables must be eliminated from objective row before beginning the simplex. Remembering that M represents a very large number, solve the transformed problem by the Simplex method.

Remark 1. *If all artificial variables in the optimal solution equal zero, the solution is optimal. If any artificial variables are positive in the optimal solution, the problem is infeasible.*

Example 3: (The Bevco Problem, pages 172–177, Winston text) Bevco manufactures an orange-flavored soft drink called Oranj by combining orange soda and orange juice. Each orange soda contains 0.5 oz of sugar and 1 mg of vitamin C. Each ounce of orange juice contains 0.25 oz of sugar and 3 mg of vitamin C. It costs Bevco 2¢ to produce an ounce of orange soda and 3¢ to produce an ounce of orange juice. Bevco's marketing department has decided that each 10-oz bottle of Oranj must contain at least 30 mg of vitamin C and at most 4 oz of sugar. Use linear programming to determine how Bevco can meet the marketing department's requirements at minimum cost.

Let x_1 number of ounces of orange soda in a bottle of Oranj and x_2 be the number of ounces of orange juice in a bottle of Oranj. Then, the linear programming problem can be written as

$$\begin{aligned} z = 2x_1 + 3x_2 &\to \min \\ 0.5x_1 + 0.25x_2 &\leq 4 && \text{sugar constraint} \\ x_1 + 3x_2 &\geq 20 && \text{vitamin C constraint} \\ x_1 + x_2 &= 10 && \text{10 oz in one bottle of Oranj} \\ x_1 \geq 0, x_2 &\geq 0 \end{aligned}$$

Next, the problem has to be converted to the canonical form. Convert the objective function $z' = -2x_1 - 3x_2 \to \max$, where $z' = -z$. After introducing the slack s_1 and surplus s_2 variables the problem can be expressed as

$$\begin{aligned} z' = -2x_1 - 3x_2 &\to \max \\ 0.5x_1 + 0.25x_2 + s_1 &= 4 \\ x_1 + 3x_2 - s_2 &= 20 \\ x_1 + x_2 &= 10 \\ x_1 \geq 0, x_2 &\geq 0 \\ s_1 \geq 0, s_2 &\geq 0 \end{aligned}$$

The linear programming problem in canonical form has z' and s_1 which could be used for constructing the initial simplex table, but row 3 would violate sign restriction. In order to use the Simplex method, a basic feasible solution is needed. To remedy the predicament, artificial variables are created.

$$
\begin{aligned}
z' = -2x_1 - 3x_2 &\to \max \\
0.5x_1 + 0.25x_2 + s_1 &= 4 \\
x_1 + 3x_2 - s_2 + y_1 &= 20 \\
x_1 + x_2 + y_2 &= 10
\end{aligned}
$$

In the optimal solution, all artificial variables must be set equal to zero. To accomplish this, a term $-My_i$ has to be added to the objective function for each artificial variable y_i. M represents some very large number. The modified canonical form then becomes

$$
\begin{aligned}
z' + 2x_1 + 3x_2 + My_1 + My_2 &= 0 \\
0.5x_1 + 0.25x_2 + s_1 &= 4 \\
x_1 + 3x_2 - s_2 + y_1 &= 20 \\
x_1 + x_2 + y_2 &= 10
\end{aligned}
$$

Modifying the objective function this way makes it extremely costly for an artificial variable to be positive. The optimal solution should force $y_1 = y_2 = 0$. Finally, the initial simplex table for this problem is

x_1	x_2	s_1	s_2	y_1	y_2	z'	b	variables	ratio
$\frac{1}{2}$	$\frac{1}{4}$	1	0	0	0	0	4	s_1	
1	3	0	-1	1	0	0	20	y_1	
1	1	0	0	0	1	0	10	y_2	
2	3	0	0	M	M	1	0	z	

Since we must eliminate all artificial variables from the objective row, perform the following operation $R_4 - MR_2 - MR_3$.

x_1	x_2	s_1	s_2	y_1	y_2	z'	b	variables	ratio
$\frac{1}{2}$	$\frac{1}{4}$	1	0	0	0	0	4	s_1	16
1	**3**	0	-1	1	0	0	20	y_1	$\frac{20}{3} \approx 6.67$
1	1	0	0	0	1	0	10	y_2	10
$2-2M$	$3-4M$	0	M	0	0	1	$-30M$	z	

Now, we can see that the second column is the pivot column, because it contains the smallest entry in the objective row. The pivot row is obtained by choosing the smallest non-negative ration among all rows. Thus, the second row is the pivot row. Diving R_2 by 3 and applying the operations $R_1 - \frac{1}{4}R_2$, $R_3 - R_2$, and $R_4 - (3-4M)R_2$, we get the second simplex table as

x_1	x_2	s_1	s_2	y_1	y_2	z'	b	variables	ratio
$\frac{5}{12}$	0	1	$\frac{1}{12}$	$-\frac{1}{12}$	0	0	$\frac{7}{3}$	s_1	$\frac{28}{5} = 5.6$
$\frac{1}{3}$	1	0	$-\frac{1}{3}$	$\frac{1}{3}$	0	0	$\frac{20}{3}$	x_2	20
$\mathbf{\frac{2}{3}}$	0	0	$\frac{1}{3}$	$-\frac{1}{3}$	1	0	$\frac{10}{3}$	y_2	5
$1 - \frac{2}{3}M$	0	0	$1 - \frac{1}{3}M$	$-1 + \frac{4}{3}M$	0	1	$-20 - \frac{10}{3}M$	z	

Applying the same procedure as described above, we determine the next pivot element to be $\frac{2}{3}$ in the first column and third row. Dividing R_3 by $\frac{2}{3}$ and applying the operations $R_1 - \frac{5}{12}R_3$, $R_2 - \frac{1}{3}R_3$, and $R_4 - (1 - \frac{2}{3}M)R_3$, we get the third simplex table as

x_1	x_2	s_1	s_2	y_1	y_2	z'	b	variables
0	0	1	$-\frac{1}{8}$	$\frac{1}{8}$	$-\frac{5}{8}$	0	$\frac{1}{4}$	s_1
0	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	5	x_2
1	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$	0	5	x_1
0	0	0	$\frac{1}{2}$	$-\frac{1}{2} + M$	$-\frac{3}{2} + M$	1	-25	z

In the above table we see that there are no negative entries in the objective row. Hence, the optimal solution is found. Therefore, $z_{\max} = -25$ for $x_1 = 5, x_2 = 5, s_1 = \frac{1}{4}, s_2 = 0$. Hence, the solution of the original problem is $z_{\min}(5, 5, \frac{1}{4}, 0) = 25$.

Solutions: special cases

Example 4: (**Alternate optimal solutions**) Consider the following linear programming problem

$$\begin{aligned} z = x_1 + 0.5x_2 \to \max \\ 2x_1 + x_2 \le 4 \\ x_1 + 2x_2 \le 3 \\ x_1 \ge 0, x_2 \ge 0 \end{aligned}$$

As before, we add slack variables s_1, s_2 and solve the problem by the Simplex method, using table representation.

x_1	x_2	s_1	s_2	z	b	variables	ratio
2	1	1	0	0	4	s_1	$\frac{4}{2} = 2$
1	2	0	1	0	3	s_2	$\frac{3}{1} = 3$
-1	$-\frac{1}{2}$	0	0	1	0	z	
1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	2	x_1	
0	$\frac{3}{2}$	$-\frac{1}{2}$	1	0	1	s_2	
0	0	$\frac{1}{2}$	0	1	2	z	

Now, we may see that this is an optimal solution. Interestingly, the coefficient of the nonbasic variable x_2 in the objective row happens to be equal to 0. However, if we increase x_2 (from its current value of 0), this will not effect the value of z. Increasing x_2 produces changes in the other variables, of course, through the equations in rows 1 and 2. In fact, we can pivot to get a different basic solution with the same objective value $z = 2$.

x_1	x_2	s_1	s_2	z	b	variables
1	0	$\frac{2}{3}$	$-\frac{1}{3}$	0	$\frac{5}{3}$	x_1
0	1	$-\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{2}{3}$	x_2
0	0	$\frac{1}{2}$	0	1	2	z

Therefore, the optimal solutions can be represented as $z_{\max}(x) = 2$, where $x = \alpha p_1 + (1-\alpha)p_2 = \left(\frac{1}{3}\alpha + \frac{5}{3}, (1-\alpha)\frac{2}{3}, 0, \alpha\right)$ with $p_1 = (2, 0, 0, 1)$, $p_2 = (\frac{5}{3}, \frac{2}{3}, 0, 0)$ and $0 \leq \alpha \leq 1$.

Remark 2. *The linear programming problem has alternative optimal solutions (multiple optimal solutions) if, at least, one of the coefficients of the nonbasic variable in the objective row equals to zero.*

Example 5: (**Degeneracy**) Consider the following linear programming problem

$$\begin{aligned} z = 2x_1 + x_2 &\to \max \\ 3x_1 + x_2 &\leq 6 \\ x_1 - x_2 &\leq 2 \\ x_2 &\leq 3 \\ x_1 \geq 0, x_2 &\geq 0 \end{aligned}$$

The initial simplex table can be written in the following form

x_1	x_2	s_1	s_2	s_3	z	b	variables	ratio
3	1	1	0	0	0	6	s_1	$\frac{6}{3} = 2$
1	−1	0	1	0	0	2	s_2	$\frac{2}{1} = 2$
0	1	0	0	1	0	3	s_3	
−2	−1	0	0	0	1	0	z	

We may see that the first column can be chosen as the pivot column and the second row as the pivot row (the minimum ratio is a tie, and ties are broken arbitrarily). After a simple calculations this yields the second table below.

x_1	x_2	s_1	s_2	s_3	z	b	variables	ratio
0	**4**	1	−3	0	0	0	s_1	$\frac{0}{4}$
1	−1	0	1	0	0	2	x_1	
0	1	0	0	1	0	3	s_3	$\frac{3}{1}$
0	−3	0	2	0	1	4	z	

Note that this basic solution has a basic variable (namely s_1) which is equal to zero. When this occurs, we say that the basic solution is degenerate. Let us continue the steps of the Simplex method. Next, we have that the second column is the pivot column. After that, we calculate the ratios and may see that the minimum ratio occurs in the first row. So let us perform the corresponding pivot.

x_1	x_2	s_1	s_2	s_3	z	b	variables	ratio
0	1	$\frac{1}{4}$	$-\frac{3}{4}$	0	0	0	x_2	
1	0	$\frac{1}{4}$	$\frac{1}{4}$	0	0	2	x_1	8
0	0	$-\frac{1}{4}$	$\mathbf{\frac{3}{4}}$	1	0	3	s_3	4
0	0	$\frac{3}{4}$	$-\frac{1}{4}$	0	1	4	z	

We get exactly the same solution. The only difference is that we have interchanged the names of a nonbasic variable with that of a degenerate basic variable (x_2 and s_1). However, we see that this solution is not optimal. Now, the fourth column is the pivot column and the third row. After pivoting, we get the following table

x_1	x_2	s_1	s_2	s_3	z	b	variables
0	1	0	0	1	0	3	x_2
1	0	$\frac{1}{3}$	0	$-\frac{1}{3}$	0	1	x_1
0	0	$-\frac{1}{3}$	1	$\frac{4}{3}$	0	4	s_2
0	0	$\frac{2}{3}$	0	$\frac{1}{3}$	1	5	z

In the above table we see that there are no negative entries in the objective row. Hence, the optimal solution is found. So, after all, degeneracy did not prevent the Simplex method to find the optimal solution $z_{\max}(1, 3) = 5$. It just slowed things down a little. Unfortunately, on other examples, degeneracy may lead to *cycling*, i.e. a sequence of pivots that goes through the same tables and repeats itself indefinitely. In theory, cycling can be avoided by choosing the entering variable (pivot column) with smallest index among all those with a negative coefficient in the objective row, and by breaking ties in the minimum ratio test by choosing the leaving variable with smallest index (this is known as Bland's rule). This rule, although it guaranties that cycling will never occur, turns out to be somewhat inefficient. Actually, in commercial codes, no effort is made to avoid cycling. This may come as a surprise, since degeneracy is a frequent occurrence. But there are two reasons for this:

- Although degeneracy is frequent, cycling is extremely rare.

- The precision of computer arithmetic takes care of cycling by itself: round off errors accumulate and eventually gets the method out of cycling.

Example 6: (**Unbounded optimum**) Consider the following linear programming problem

$$z = 2x_1 + x_2 \to \max$$
$$-x_1 + x_2 \leq 1$$
$$x_1 - 2x_2 \leq 2$$
$$x_1 \geq 0, x_2 \geq 0$$

Solving by the simplex method, we get

x_1	x_2	s_1	s_2	z	b	variables
-1	1	1	0	0	1	s_1
1	-2	0	1	0	2	s_2
-2	-1	0	0	1	0	z
0	-1	1	1	0	3	s_1
1	-2	0	1	0	2	x_1
0	-5	0	2	1	4	z

At this stage, we have to choose the second column as a pivot column, but there is no ratio to compute, since all entries are negative. As we start increasing x_2, the value of z increases and the values of the basic variables increase as well. There is nothing to stop them going off to infinity. Therefore, the problem is unbounded.

Summary

Let us briefly summarize the presented above theory. Consider a linear programming problem given in the standard form

$$z = CX \to \max$$
$$AX \leq B$$
$$X \geq 0$$

It has to be converted to the canonical form

$$z = CX \to \max$$
$$AX = B$$
$$X \geq 0$$

- If it is necessary to find $z \to \min$, then we can use the substitution $z' = -z$ and solve the maximum problem $z' \to \max$.
- If there are inequalities $\leq, \geq$ involved in the constraints, then the equality sign $=$ can be obtained using slack or surplus variables, i.e.
  - if $a_1x_1 + a_2x_2 \leq c$, then $a_1x_1 + a_2x_2 + s_1 = c$ for $s_1 \geq 0$;

– if $a_1x_1 + a_2x_2 \geq c$, then $a_1x_1 + a_2x_2 - s_2 = c$ for $s_2 \geq 0$.

- If, for example, there is no condition $x_1 \geq 0$, then $x_1 = x_1' - x_1''$ for $x_1', x_1'' \geq 0$.

The Simplex method can be applied if:

1. $B \geq 0$;
2. there is a basic feasible solution, i.e. it is possible to single out an identity matrix of order m in the initial simplex table. If there is no basic feasible solution, then so-called M method can be applied to construct initial simplex table.

Definition 1. *The variables corresponding to the columns of the identity matrix in the initial simplex table are called* ***basic variables*** *while the remaining variables are called nonbasic or* ***free variables****.*

Theorem 1. *A basic feasible solution to a linear programming problem corresponds to an extreme point in the convex set of feasible solutions.*

Corollary 1. *Each extreme point corresponds to one or more basic feasible solutions. If one of the basic feasible solutions is non-degenerate, then an extreme point corresponds to it uniquely.*

General structure of simplex table:

Variables												
basic			slack/surplus			artificial					basis	
x_1	$\dots$	x_n	s_1	$\dots$	s_k	y_1	$\dots$	y_l	z	b	variables	ratio

Summary of the Simplex method:

Step 1: Add slack variables to change the constraints into equations and write all variables to the left of the equal sign and constants to the right.

Step 2: Write the objective function with all nonzero terms to the left of the equal sign and zero to the right. The variable to be maximized must be positive.

Step 3: Set up the initial simplex tableau by creating an augmented matrix from the equations, placing the equation for the objective function last.

Step 4: Determine a pivot element and use matrix row operations to convert the column containing the pivot element into a unit column.

Step 5: If negative elements still exist in the objective row, repeat **Step 4**. If all elements in the objective row are positive, the process has been completed.

Step 6: When the final matrix has been obtained, determine the final basic solution. This will give the maximum value for the objective function and the values of the variables where this maximum occurs.

Geometric interpretation of the Simplex method: The simplex method always starts at the origin (which is a corner point) and then jumps from a corner point to the neighboring corner point until it reaches the optimal corner point (if bounded). Therefore, at each one of the simplex iterations, we are searching for a better solution among the vertices of a Simplex.

Exercises

Solve the following linear programming problem by Simplex method.

Example 7:

$$
\begin{aligned}
z = 6x_1 - 8x_2 + x_3 \to \max \\
3x_1 + x_2 \le 10 \\
4x_1 - x_2 \le 5 \\
x_1 + x_2 - x_3 \ge -3 \\
x_1 \ge 0, x_2 \ge 0, x_3 \ge 0
\end{aligned}
$$

Example 8:

$$
\begin{aligned}
z = -2x_1 + 3x_2 - 6x_3 - x_4 \to \min \\
2x_1 + x_2 - 2x_3 + x_4 = 24 \\
x_1 + 2x_2 + 4x_3 \le 22 \\
x_1 - x_2 + 2x_3 \ge 10 \\
x_1 \ge 0, x_2 \ge 0, x_3 \ge 0, x_4 \ge 0
\end{aligned}
$$

Example 9:

$$
\begin{aligned}
z = 12x_1 + 14x_2 + 16x_3 \to \max \\
x_1 + x_2 + x_3 \le 24 \\
x_1 + 2x_2 + 3x_3 \ge 51 \\
3x_1 + 2x_2 + x_3 = 57 \\
x_1 \ge 0, x_2 \ge 0, x_3 \ge 0
\end{aligned}
$$

Problems

Solve the following linear programming problem by Simplex method.

4.1:

$$
\begin{aligned}
z = 4x_1 + 3x_2 \to \max \\
3x_1 + x_2 \le 9 \\
-x_1 + x_2 \le 1 \\
x_1 + x_2 \le 6 \\
x_1 \ge 0, x_2 \ge 0
\end{aligned}
$$

4.2:

$$
\begin{aligned}
z = 2x_1 - x_4 \to \max \\
x_1 + x_2 = 20 \\
x_2 + 2x_4 \geq 5 \\
-x_1 + x_2 + x_3 \leq 8 \\
x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0
\end{aligned}
$$

4.3:

$$
\begin{aligned}
z = 2x_1 - 6x_2 + 5x_5 \to \max \\
-2x_1 + x_2 + x_3 + x_5 = 20 \\
3x_1 - x_2 + x_4 + 3x_5 = 24 \\
3x_1 - x_2 - 12x_5 + x_6 = 18 \\
x_i \geq 0 \qquad \text{for } i = 1, \ldots, 6
\end{aligned}
$$

4.4:

$$
\begin{aligned}
z = 2x_1 + x_2 - x_3 + x_4 - x_5 \to \max \\
x_1 + x_2 + x_3 = 5 \\
2x_1 + x_2 + x_4 = 9 \\
x_1 + 2x_2 + x_5 = 7 \\
x_i \geq 0 \qquad \text{for } i = 1, \ldots, 5
\end{aligned}
$$

4.5:

$$
\begin{aligned}
z = 3x_1 + 2x_2 - x_3 \to \max \\
x_1 + 3x_2 + x_3 \leq 5 \\
2x_1 + 3x_2 - x_3 \geq 2 \\
3x_1 - 2x_2 + x_3 \geq 5 \\
x_1 \geq 0, x_2 \geq 0, x_3 \geq 0
\end{aligned}
$$

4.6:

$$
\begin{aligned}
z = 2x_1 - x_2 + x_3 \to \max \\
x_1 + x_2 - 3x_3 \leq 8 \\
4x_1 - x_2 + x_3 \geq 2 \\
2x_1 + 3x_2 - x_3 \geq 4 \\
x_1 \geq 0, x_2 \geq 0, x_3 \geq 0
\end{aligned}
$$

4.7:

$$
\begin{aligned}
z = 3x_1 + 4x_2 \to \max \\
-x_1 - x_2 \geq -12 \\
5x_1 + 2x_2 \geq 36 \\
7x_1 + 4x_2 \geq 14 \\
x_1 \geq 0, x_2 \geq 0
\end{aligned}
$$

4.8:

$$\begin{aligned} z = 4x_1 + 2x_2 &\to \min \\ 3x_1 + x_2 &\geq 27 \\ -x_1 - x_2 &\leq -21 \\ x_1 + 2x_2 &\geq 30 \\ x_1 \geq 0, x_2 &\geq 0 \end{aligned}$$

4.9:

$$\begin{aligned} z = 9x_1 + 14x_2 + x_3 &\to \max \\ 9x_1 + 4x_2 + 4x_3 &\leq 54 \\ 9x_1 + 5x_2 + 5x_3 &\leq 63 \\ x_2 &\leq 5 \\ x_1 \geq 0, x_2 \geq 0, x_3 &\geq 0 \end{aligned}$$

4.10:

$$\begin{aligned} z = 3x_1 + 2x_2 + x_3 &\to \max \\ 2x_1 + 2x_2 + 4x_3 &\leq 540 \\ x_1 + 5x_2 + x_3 &\leq 360 \\ 6x_1 + 2x_2 + x_3 &\leq 180 \\ x_1 \geq 0, x_2 \geq 0, x_3 &\geq 0 \end{aligned}$$

Answers to problems

1. $z_{\max}(2,3) = 17$.
2. $z_{\max}(20,0,0,\frac{5}{2}) = \frac{75}{2}$.
3. $z_{\max}(0,0,12,0,0,8,114) = 40$.
4. $z_{\max}(3,2,0,1,0) = 9$.
5. $z_{\max}(9,0,0) = 27$.
6. The problem is unbounded and there is no solution.
7. $z_{\max}(4,8) = 44$.
8. $z_{\min}(3,18) = 48$.
9. $z_{\max}(x) = 108$, where $x = \left(2\alpha, 5, \frac{38-18\alpha}{5}, \frac{18}{5}(1-\alpha), 0, 0\right)$ and $0 \le \alpha \le 1$.
10. $z_{\max}(x) = 180$, where $x = (0, 30(1+\alpha), 60(2-\alpha), 180\alpha, 30(1-\alpha), 0)$ and $0 \le \alpha \le 1$.