ISS0031 Modeling and Identification

Lecture 3

Convex set and function

Let $S \neq \emptyset, S \subset \mathbb{R}^n$ and $x_1, x_2 \in S$.

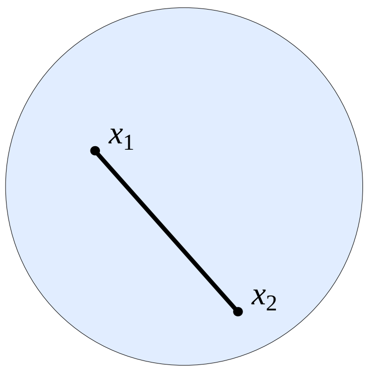
Definition 1. *The set* $[x_1, x_2] = \{x | x = \lambda x_1 + (1 - \lambda)x_2, 0 \leq \lambda \leq 1\}$ *is called a line segment with the endpoints* x_1, x_2.

Example 1: Let $x_1(2, 1)$ and $x_2(4, 3)$. Next, using the formula from Definition 1, we get

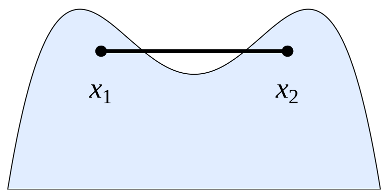
$$
\begin{aligned}
x_1 &= 2\lambda + (1 - \lambda)4 \\
x_2 &= \lambda + (1 - \lambda)3 \\
0 &\leq \lambda \leq 1
\end{aligned}
$$

Definition 2. *A set* S *in a vector space over* $\mathbb{R}$ *is called a convex set if the line segment joining any pair of points* $x_1, x_2 \in S$ *lies entirely in* S.

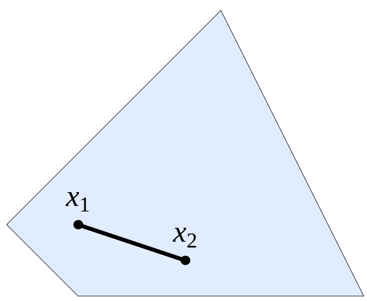
Example 2: Consider different examples of convex and non-convex sets.



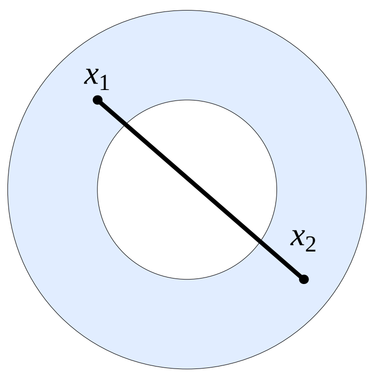
(a) convex



(b) non-convex



(c) convex



(d) non-convex

Proposition 1. *A solution set $\mathbb{L}$ for the linear inequality $a_1x_1+a_2x_2+\cdots+a_nx_n \le b$ is a convex set.*

Proof: Let the points $C_1(c_1^1, c_2^1, \ldots, c_n^1)$ and $C_2(c_1^2, c_2^2, \ldots, c_n^2)$ be solutions of the given inequality. Then,

$$a_1c_1^1 + a_2c_2^1 + \cdots + a_nc_n^1 \le b$$
$$a_1c_1^2 + a_2c_2^2 + \cdots + a_nc_n^2 \le b$$

Next, we multiply the first inequality by λ, the second inequality by $1-\lambda$ and add results

$$a_1(\lambda c_1^1 + (1-\lambda)c_1^2) + a_2(\lambda c_2^1 + (1-\lambda)c_2^2) + \cdots \\ + a_n(\lambda c_n^1 + (1-\lambda)c_n^2) \le \lambda b + (1-\lambda)b = b.$$

Using the obtained result, we can conclude that the point $C(\lambda c_1^1 + (1-\lambda)c_1^2, \lambda c_2^1 + (1-\lambda)c_2^2, \ldots, \lambda c_n^1 + (1-\lambda)c_n^2) = \lambda C_1 + (1-\lambda)C_2 \in \mathbb{L}$. ■

Proposition 2. *The intersection of any finite number of convex sets is a convex set.*

Proof: Suppose $S_1, S_2, \ldots, S_n$ are convex sets. Then their intersection $\bigcap_{i=1}^n S_i = \{x : x \in S_i \forall i = 1, \ldots, n\}$ is also a convex set. To see this, consider $x_1, x_2 \in \bigcap_{i=1}^n S_i$ and $0 \le \lambda \le 1$. Then, $\lambda x_1 + (1-\lambda)x_2 \in S_i$ for $i = 1, \ldots, n$ by Definition 2. Therefore, $\lambda x_1 + (1-\theta)x_2 \in \bigcap_{i=1}^n S_i$. ■

Corollary 1. *The solution set of a system of linear inequalities is a convex set.*

Corollary 2. *The solution set of linear equations is a convex set.*

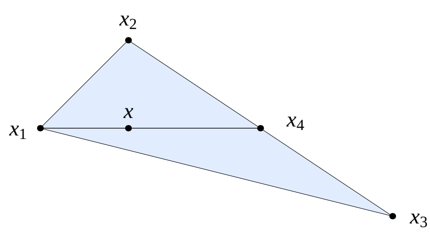
Corollary 3. *The solution set of constraints for linear programming problem (set of feasible solutions) is a convex set.*

Next, we present the generalization of Definition 2.

Definition 3. *Given a finite number of points $x_1, x_2, \ldots, x_n$ in a real vector space, a convex combination of these points is a point of the form $\alpha_1x_1+\alpha_2x_2+\cdots+\alpha_nx_n$, where the real numbers $\alpha_i \ge 0$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$.*

Example 3: Consider two special cases of Definition 3:
Case 1: Let $n = 2$, then a convex combination of points x_1, x_2 is in the form $x = \lambda_1x_1+\lambda_2x_2$, where $\lambda_1+\lambda_2 = 1$ and $\lambda_1, \lambda_2 \ge 0$. Denote by $\lambda_1 = \lambda$, then $\lambda_2 = 1-\lambda$ and we get that the convex combination of two points is $x = \lambda x_1 + (1-\lambda)x_2$.

Case 2: Let $n = 3$, then $x_4 = \alpha_2x_2 + \alpha_3x_3$, where $\alpha_2 + \lambda_3 = 1$ and $\alpha_2, \alpha_3 \ge 0$; $x = \beta_1x_1 + \beta_4x_4$, where $\beta_1 + \beta_4 = 1$ and $\beta_1, \beta_4 \ge 0$. Then, we get that the convex combination of 3 points is: $x = \beta_1x_1+\beta_4(\alpha_2x_2+\alpha_3x_3) = \beta_1x_1+\beta_4\alpha_2x_2+\beta_4\alpha_3x_3 = \lambda_1x_1+\lambda_2x_2+\lambda_3x_3$, where $\lambda_1+\lambda_2+\lambda_3 = \beta_1+(\beta_4\alpha_2+\beta_4\alpha_3) = \beta_1 + \beta_4(\alpha_2 + \alpha_3) = \beta_1 + \beta_4 = 1$.



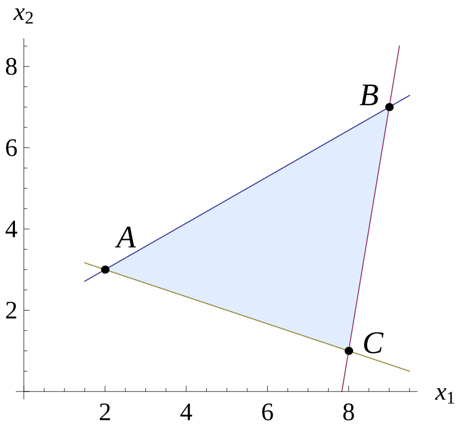
Definition 4. *Let x be a convex combination of points from the set S. Then, S is called convex if $x \in S$.*

Example 4: Verify that the point $P(6,3)$ is an interior point of the set

$$\begin{aligned} -4x_1 + 7x_2 &\leq 13 \\ 6x_1 - x_2 &\leq 47 \\ x_1 + 3x_2 &\geq 11 \end{aligned}$$

and express P as a convex combination of the vertices of the solutions of these system.

Substituting coordinates of the point P to each inequality, one can see that all of them hold. Therefore, the point $P(6,3)$ is the interior point of the corresponding polytope. Draw the graphs of given inequalities as follows.



One may easily see that the obtained polytope has 3 vertices. In order to find coordinates of A, B and C, we have to solve 3 systems of linear equation. Let us find coordinates of the point A. For that purpose we have to solve the following system of linear equations.

$$\begin{aligned} -4x_1 + 7x_2 &= 13 \\ x_1 + 3x_2 &= 11 \end{aligned}$$

One method for solving such a system is as follows. First, solve the second equation for x_1 in terms of x_2 as $x_1 = 11 - 3x_2$. Now, substitute this expression for x_1 into the first equation as $-4(11 - 3x_2) + 7x_2 = 13$. This results in a single equation involving only the variable x_2. Solving gives $x_2 = 3$, and substituting this into the equation for x_1 yields $x_1 = 2$. Therefore, $A(2,3)$. Similarly, we can calculate that $B(9,7)$ and $C(8,1)$. Next, according to Definition 3, we get $X = \alpha A + \beta B + \gamma C$ with $\alpha + \beta + \gamma = 1$ and $\alpha, \beta, \gamma \geq 0$. Thus, we can construct the following system of equations.

$$\begin{aligned} 2\alpha + 9\beta + 8\gamma &= 6 \\ 3\alpha + 7\beta + \gamma &= 3 \\ \alpha + \beta + \gamma &= 1 \end{aligned}$$

A solution to the system above is given by $\alpha = 7/19, \beta = 8/19, \gamma = 4/19$. Finally, substituting the obtained solution to the expression for X, we get

$$X = \frac{7}{19}A + \frac{4}{19}B + \frac{8}{19}C.$$

Definition 5. *A real valued function $f : S \to \mathbb{R}$ defined on a convex set S in a vector space is called convex or concave if, for any two points x_1 and x_2 in S and any $0 \le \lambda \le 1$, $f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda)f(x_2)$ or $f(\lambda x_1 + (1-\lambda)x_2) \ge \lambda f(x_1) + (1-\lambda)f(x_2)$.*

Example 5: $f(x) = x^2$ is convex on $\mathbb{R}$; $f(x) = \log x$ is concave on $\mathbb{R}^+$; $f(x) = \frac{1}{x}$ is convex on $\mathbb{R}^+$ and concave on $\mathbb{R}^-$; $f(x) = x^3 - x$ is neither convex nor concave on $\mathbb{R}$.

Proposition 3. *A linear function $f = a_1x_1 + a_2x_2 + \cdots + a_nx_n$ is both convex and concave.*

Proof: Let $X_1(x_1', x_2', \ldots, x_n')$ and $X_2(x_1'', x_2'', \ldots, x_n'')$. Then, $\lambda X_1 + (1-\lambda)X_2 = (\lambda x_1' + (1-\lambda)x_1'', \ldots, \lambda x_n' + (1-\lambda)x_n'')$ and

$$\begin{aligned} f(\lambda X_1 + (1-\lambda)X_2) &= a_1(\lambda x_1' + (1-\lambda)x_1'') + \cdots + a_n(\lambda x_n' + (1-\lambda)x_n'') = \\ &= \lambda(a_1x_1' + a_2x_2' + \cdots + a_nx_n') + (1-\lambda)(a_1x_1'' + a_2x_2'' + \cdots + a_nx_n'') = \\ &= \lambda f(X_1) + (1-\lambda)f(X_2). \end{aligned}$$

Hence, we can conclude that f is convex and concave. ■

Convex optimization problem

Definition 6. *A function $f(x)$ is said to have a local maximum (minium) at x_0 if there exists an interval I around x_0 such that $f(x_0) \ge f(x)$ ($f(x_0) \le f(x)$) for all $x \in I$.*

Definition 7. *We say that the function $f(x)$ has a global maximum (minimum) at $x = x_0$ on the interval I, if $f(x_0) \ge f(x)$ ($f(x_0) \le f(x)$) for all $x \in I$.*

Note that if $f(x)$ is a continuous function on a closed bounded interval $[a, b]$, then $f(x)$ will have a global maximum and a global minimum on $[a, b]$. On the other hand, if the interval is not bounded or closed, then there is no guarantee that a continuous function $f(x)$ will have global extremum.

Example 6: $f(x) = x^2$ does not have a global maximum on the interval $[0, \infty)$, the function $f(x) = -\frac{1}{x}$ does not have a global minimum on the interval $(0, 1)$.

Definition 8. *A convex optimization problem is a problem where all of the constraints are convex functions, and the objective is a convex function if minimizing, or a concave function if maximizing.*

Theorem 1. *For a convex optimization problem all locally optimal points are globally optimal.*

Linear programming as a special case of convex optimization problem

The linear programming problem can be stated as follows:

$$z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \min$$

subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\le b_1 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\le b_m \end{aligned}$$

and

$$x_1 \ge 0, x_2 \ge 0, \ldots, x_n \ge 0.$$

Theorem 2. *If a linear programming problem has a solution, then it must occur at a vertex, or corner point, of the feasible set S, associated with the problem. Furthermore, if the objective function z is optimized at two adjacent vertices of S, then it is optimized at every point on the line segment joining these two vertices, in which case there are infinitely many solutions to the problem.*

Proof: The proof is by contradiction. Suppose that the optimal solution x^* is an interior point of the feasible set S. Since the set is convex, then there exist two points $x_1, x_2 \in S$ such that $x^* \in [x_1, x_2]$, i.e. $x^* = \lambda x_1 + (1-\lambda)x_2$. We know that x^* is optimal solution, then denoting $f(x) := c_1x_1 + \cdots + c_nx_n$, we get

$$\begin{aligned} f(x^*) &\ge f(x_1) \\ f(x^*) &\ge f(x_2) \end{aligned} \tag{1}$$

Since $f(x)$ is linear (convex) function

$$f(x^*) = f(\lambda x_1 + (1 - \lambda x_2)) = \lambda f(x_1) + (1-\lambda)f(x_2). \tag{2}$$

Substituting (2) to (1), we get

$$\begin{aligned} f(x_1) &\le \lambda f(x_1) + (1-\lambda)f(x_2) \\ f(x_2) &\le \lambda f(x_1) + (1-\lambda)f(x_2) \end{aligned}$$

or after simple transformations $f(x_1) = f(x_2)$. From (2) it follows that $f(x^*) = \lambda f(x_1) + (1-\lambda)f(x_2) = f(x_1)$. Hence, we get $f(x_1) = f(x_2) = f(x^*) = z_0 \in \mathbb{R}$. As a result, points x_1, x_2, x^* are in the hyperplane $f(x) = z_0$. We know that the point x^* defines this hyperplane; however, the end points of the line segment $[x_1, x_2]$ are free to choose. Therefore, points x_1, x_2 may not necessarily belong to this hyperplane. This contradicts our assumption, showing that x^* has to be on the boundary of the set S. ∎

Remark 1. *Theorem 2 tells us that our search for the solution(s) to a linear programming problem may be restricted to the examination of the set of vertices of the feasible set S associated with the problem. Since a feasible set S has finitely many vertices, the theorem suggest that the solution(s) may be found by inspecting the values of the objective function z at these vertices.*

Problems

3.1: Find intervals where the following functions are convex (concave): $f(x) = x^2$, $f(x) = e^x$, $f(x) = x^3$, $f(x) = \frac{1}{x}$, $f(x) = \frac{1}{x^2}$, $f(x) = \sin x$, $f(x) = x^5 + 5x - 6$, $f(x) = (x+1)^2(x-2)$, and $f(x) = xe^x$.

Answers to problems

1. Use the following theorem

   Theorem 3. *If $f(x)$ has a positive (negative) second derivative $f''(x)$ everywhere on $I \subseteq \mathbb{R}$, then $f(x)$ is convex (concave) on I.*