# Lecture 11: Algebraic Methods for Nonlinear Control Systems

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# Overview of the talk

- Algebraic framework: basic definitions and constructions
- Polynomial framework
- One nonlinear control problem: Realizability
- Concluding remarks

# Two common theories to study nonlinear control systems

- Differential geometrical approach: appeared in the 1970s
  A. Isidori, H. Nijmeijer, W. Respondek, A. van der Schaft, etc.
- **Differential algebraic** methods: start from the second half of 1980s
  G. Conte, M. Fliess, Ü. Kotta, C. H. Moog, A. M. Perdon, etc.

Differential Algebra

Calculus and Topology:
Ordinary differentiation and exterior derivative

Algebra:
rings, fields, etc.

# Basic definitions: Calculus

### Definition (Differentiability)

A real function is said to be differentiable at a point if its derivative exists at that point.

### Definition (Derivative)

The derivative of a function $f(x)$ with respect to the variable $x$ is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

### Proposition

*If $f(x)$ is differentiable at a point $x_0$ in $f$ is continuous at $x_0$.*

**Example:** Function $f(x) = |x|$ is continuous at 0, but not differentiable.

# Analytic and meromorphic functions

## Definition

Analytic function $f(x)$ is an infinitely differentiable function such that the Taylor series at any point $x_0$ in its domain $D$ converges to $f(x)$ for $x$ in a neighborhood of $x_0$ point-wise (and uniformly).

**Examples:** polynomial functions $f(x) = x^2 - 3x + 1$, exponential function $f(x) = \mathrm{e}^x$, trigonometric functions $f_1(x) = \cos x$, $f_2(x) = \tanh(3x)$.

## Definition

If $I$ is an open subset and $f$ is a function defined and analytic in $I$ except for poles, then $f$ is a meromorphic function on $I$.

**Examples:** rational functions $f(x) = \frac{x^2-1}{x^3+2x-1}$, Gamma function $\Gamma(z) = \int_0^\infty t^{z-1}\mathrm{e}^{-t}\mathrm{d}t$, Riemann Zeta function $\zeta(s) = \sum_{k=1}^\infty k^{-s}$.

$$\text{Analytic functions} \subset \text{Smooth functions } (C^\infty)$$

# Analytic functions: more details

## Definition

Let $I \subseteq \mathbb{R}$ be an open interval. A function $f : I \to \mathbb{R}$ is analytic at a point $x_0 \in I$ if it admits a Taylor series expansion in a neighborhood of $x_0$. If $f$ is analytic at every point of $I \subseteq \mathbb{R}$, we say that $f$ is analytic in $I$.

## Proposition

*Let $I \subseteq \mathbb{R}$ be an open interval, and let $f : I \to \mathbb{R}$ be an analytic function on $I$, then either*

1. $f \equiv 0$ *in* $I$*, or*
2. *the zeros of* $f$ *in* $I$ *are isolated.*

# Non-analytic functions: Illustrative example

The function $f(x)$ defined by

$$f(x) = \begin{cases} \sin(1/x), & \text{if} \quad x \neq 0 \\ 0, & \text{if} \quad x = 0 \end{cases}$$

is not analytic because the point $x = 0$ is a point of accumulation for the zeros of $f$.

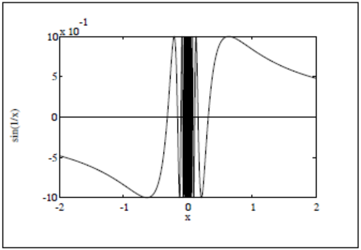



**Fig. 1.2.** Graph of $\sin(1/x)$

# Basic algebraic structures

A ring is a set $\mathcal{R}$ together with two binary operators $+$ and $*$ satisfying conditions:

1. Additive associativity: For all $a, b, c \in \mathcal{R}$, $(a+b)+c = a+(b+c)$;
2. Additive commutativity: For all $a, b \in \mathcal{R}$, $a+b = b+a$;
3. Additive identity: There exists an element $0 \in \mathcal{R}$ such that for all $a \in \mathcal{R}$, $0+a = a+0 = a$;
4. Additive inverse: For every $a \in \mathcal{R}$ there exists $-a \in \mathcal{R}$ such that $a+(-a) = (-a)+a = 0$;
5. Left and right distributivity: For all $a, b, c \in \mathcal{R}$, $a*(b+c) = (a*b)+(a*c)$ and $(b+c)*a = (b*a)+(c*a)$;
6. Multiplicative associativity: For all $a, b, c \in \mathcal{R}$, $(a*b)*c = a*(b*c)$ (a ring satisfying this property is sometimes explicitly termed an **associative ring)**;
7. Multiplicative commutativity: For all $a, b \in \mathcal{R}$, $a*b = b*a$ (a ring satisfying this property is termed a **commutative ring**);
8. Multiplicative identity: There exists an element $1 \in \mathcal{R}$ such that for all $a \neq 0 \in \mathcal{R}$, $1*a = a*1 = a$ (a ring satisfying this property is termed a **unit ring**, or sometimes a **ring with identity**);
9. Multiplicative inverse: For each $a \neq 0 \in \mathcal{R}$, there exists an element $a^{-1} \in \mathcal{R}$ such that for all $a \neq 0 \in \mathcal{R}$, $a*a^{-1} = a^{-1}*a = 1$, where 1 is the identity element.

# Basic algebraic structures: Summary

| Prop. # | Ring | Commutative ring | Division ring / Skew field | Field |
|---|---|---|---|---|
| 1 | ✓ | ✓ | ✓ | ✓ |
| 2 | ✓ | ✓ | ✓ | ✓ |
| 3 | ✓ | ✓ | ✓ | ✓ |
| 4 | ✓ | ✓ | ✓ | ✓ |
| 5 | ✓ | ✓ | ✓ | ✓ |
| 6 | ✗ / ✓ | ✓ | ✓ | ✓ |
| 7 | ✗ | ✓ | ✗ | ✓ |
| 8 | ✗ | ✓ | ✓ | ✓ |
| 9 | ✗ | ✗ | ✓ | ✓ |

# Input-output and state-space forms: single-input single-output systems

Notation: the first- and second-order derivatives are $\dot{\xi} := \frac{\mathrm{d}\xi}{\mathrm{d}t}$, $\ddot{\xi} := \frac{\mathrm{d}^2\xi}{\mathrm{d}t^2}$, and $\xi^{(k)} := \frac{\mathrm{d}^k\xi}{\mathrm{d}t^k}$ stands to the time derivative of an arbitrary order.

Input-output equation

$$y^{(n)} = \phi\left(y, \dot{y}, \ldots, y^{(n-1)}, u, \dot{u}, \ldots, u^{(s)}\right).$$

State equations

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x), \end{aligned}$$

$x : \mathbb{R} \to \mathcal{X} \subset \mathbb{R}^n$ is the vector of state variables,
$u : \mathbb{R} \to \mathcal{U} \subset \mathbb{R}$ is the input signals,
$y : \mathbb{R} \to \mathcal{Y} \subset \mathbb{R}$ is the output signal,
$f : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ and $h : \mathcal{X} \to \mathcal{Y}$ are meromorphic functions.

# Ring of analytic functions $\mathcal{R}$

Let $\mathcal{R}$ denote the ring of analytic functions in a finite number of variables from the set a finite number of independent system variables from the infinite set

$$\mathcal{C}_{ss} = \{x_i,\ i = 1, \ldots, n;\ \ u^{(k)},\ k \geq 0\}$$

or

$$\mathcal{C}_{io} = \{y, y^{(1)}, \ldots, y^{(n-1)},\ u^{(k)},\ k \geq 0\}.$$

$\mathcal{C}_{ss}$ is associated to the state-space form

$\mathcal{C}_{io}$ is associated to the input-output description

# Differential ring

Define a time derivative operator $\mathrm{d}/\mathrm{d}t : \mathcal{R} \to \mathcal{R}$ as

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t} x = f(x,u)}, \quad \frac{\mathrm{d}}{\mathrm{d}t} u_j^{(k)} = u_j^{(k+1)},$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \zeta\left(x, u^{(k)}\right) = \sum_{i=1}^{n} \frac{\partial \zeta}{\partial x_i} \frac{\mathrm{d}}{\mathrm{d}t} x_i + \sum_{k \geq 0} \frac{\partial \zeta}{\partial u^{(k)}} \frac{\mathrm{d}}{\mathrm{d}t} u^{(k)},$$

or as

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t} y^{(n-1)} = \phi(\cdot)}, \qquad \frac{\mathrm{d}}{\mathrm{d}t} y^{(l)} = y^{(l+1)}, \text{ for } l = 0, \ldots, n-2,$$

$$\frac{\mathrm{d}}{\mathrm{d}t} u^{(k)} = u^{(k+1)},$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \xi\left(y^{(l)}, u^{(k)}\right) = \sum_{l=0}^{n-1} \frac{\partial \xi}{\partial y^{(l)}} \frac{\mathrm{d}}{\mathrm{d}t} y^{(l)} + \sum_{k \geq 0} \frac{\partial \xi}{\partial u^{(k)}} \frac{\mathrm{d}}{\mathrm{d}t} u^{(k)}.$$

The pair $(\mathcal{R}, \mathrm{d}/\mathrm{d}t)$ forms an algebraic structure known as a **differential ring**.

# Differential ring: integral domain

A ring $D$ is called an *integral domain* if it does not contain any zero divisors.

It means that if $a$ and $b$ are two elements of $D$ such that $ab = 0$, then either $a = 0$ or $b = 0$ or both.

The ring $\mathcal{R}$ of analytic functions is **integral domain**.

# Differential ring: integral domain

## Ring of smooth functions

**Remark:** $C^\infty$ functions too form a ring, but it contains zero divisors.

**Example:** Consider two smooth functions defined as

$$f_1(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{if } x < 0, \\ 0, & \text{if } x \geq 0 \end{cases}$$

and

$$f_2(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ e^{-\frac{1}{x^2}}, & \text{if } x > 0, \end{cases}$$

whose product is identically zero.

# Field of meromorphic functions $\mathcal{K}$

## Construction: sketch

Construction:

1. Let $\mathcal{S}$ be multiplicative subset of $\mathcal{R}$.
2. Consider the set of fractions of $\mathcal{R}$ over $\mathcal{S}$, denoted as $\mathcal{K} := \mathcal{S}^{-1}\mathcal{R}$.
3. Elements of $\mathcal{K}$ are meromorphic functions of the form $\beta^{-1}\alpha$, where $\alpha \in \mathcal{R}$, $\beta \in \mathcal{S}$.
4. Since $\mathcal{R}$ is integral domain, $\mathcal{K}$ forms an algebraic structure known as a **field of fractions** (quotient field).

**General idea:** The field of fractions $\mathcal{K}$ of an integral domain $\mathcal{R}$ is the smallest field containing $\mathcal{R}$, since it is obtained from $\mathcal{R}$ by adding the least needed to make $\mathcal{R}$ a field, namely the possibility of dividing by any nonzero element.

# Differential field

The operator $\mathrm{d}/\mathrm{d}t$ can be extended so that $\mathrm{d}/\mathrm{d}t : \mathcal{K} \to \mathcal{K}$.
For $b^{-1}a \in \mathcal{K}$ we define

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(b^{-1}a\right) := (b^2)^{-1}(\dot{a}b - a\dot{b}),\ a \in \mathcal{R},\ b \in \mathcal{S}.$$

The pair $(\mathcal{K}, \mathrm{d}/\mathrm{d}t)$ is a **differential field**.

# Differential vector space $\mathcal{E}$

Consider next the infinite set of symbols

$$\mathrm{d}\mathcal{C}_{ss} = \{\mathrm{d}x_i,\ i = 1, \dots, n;\ \mathrm{d}u^{(k)},\ k \geq 0\}$$

or

$$\mathrm{d}\mathcal{C}_{io} = \{\mathrm{d}y, \mathrm{d}y^{(1)}, \dots, \mathrm{d}y^{(n-1)}\, i = 1, \dots, n;\ \mathrm{d}u^{(k)},\ k \geq 0\}$$

and denote by $\mathcal{E}$ the **differential vector space** spanned over the field $\mathcal{K}$ by the elements of $\mathrm{d}\mathcal{C}$, i.e.

$$\mathcal{E} := \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\mathcal{C}\}.$$

# Differential forms

Any element of $\mathcal{E}$ has the form

$$\omega = \sum_{i=1}^{n} \alpha_i \mathrm{d}x_i + \sum_{k \geq 0} \beta_k \mathrm{d}u^{(k)}$$

or

$$\omega = \sum_{i=1}^{n} \alpha_i \mathrm{d}y^{(i)} + \sum_{k \geq 0} \beta_k \mathrm{d}u^{(k)},$$

where $\alpha_i, \beta_k \in \mathcal{K}$ and only a finite number of coefficients $\beta_k$ are nonzero.

The elements of $\mathcal{E}$ are called the differential **one-forms**.

# Differential forms

## Operators $\mathrm{d}$ and $\mathrm{d}/\mathrm{d}t$ in $\mathcal{E}$

The differential operator $\mathrm{d}:\mathcal{K}\to\mathcal{E}$ is defined as

$$\mathrm{d}\zeta\left(x,u^{(k)}\right)=\sum_{i=1}^{n}\frac{\partial\zeta}{\partial x_i}\mathrm{d}x_i+\sum_{k\geq 0}\frac{\partial\zeta}{\partial u^{(k)}}\mathrm{d}u^{(k)}$$

or

$$\mathrm{d}\xi\left(y^{(l)},u^{(k)}\right)=\sum_{l=0}^{n-1}\frac{\partial\xi}{\partial y^{(l)}}\mathrm{d}y^{(l)}+\sum_{k\geq 0}\frac{\partial\xi}{\partial u^{(k)}}\mathrm{d}u^{(k)}.$$

For the one-form $\omega=\lambda_i\mathrm{d}\varphi_i$, where $\lambda_i\in\mathcal{K}$ and $\varphi_i\in\mathcal{C}$, the operator $\mathrm{d}/\mathrm{d}t:\mathcal{E}\to\mathcal{E}$ is defined as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{l}\lambda_l\mathrm{d}\varphi_l\right):=\sum_{l}\left(\dot{\lambda}_l\mathrm{d}\varphi_l+\lambda_l\mathrm{d}\dot{\varphi}_l\right).$$

**Remark:** Operators $\mathrm{d}$ and $\mathrm{d}/\mathrm{d}t$ commute, i.e. for $\varphi\in\mathcal{K}$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}\varphi)=\mathrm{d}\left(\frac{\mathrm{d}}{\mathrm{d}t}\varphi\right)=\mathrm{d}\dot{\varphi}.$$

# Differential forms

**Example:** Let $F = \sin(x_1 x_2) \in \mathcal{K}$. Then differentiating $F$ with respect to $x_1$ and $x_2$, we get

$$\mathrm{d}F = \cos(x_1 x_2) x_2 \mathrm{d}x_1 + \cos(x_1 x_2) x_1 \mathrm{d}x_2 = \cos(x_1 x_2)[x_2 \mathrm{d}x_1 + x_1 \mathrm{d}x_2]$$

with $\mathrm{d}F \in \mathcal{E}$.

# Differential forms

## Two-forms: exterior derivative and wedge product

Starting from the space $\mathcal{E}$ it is possible to build up the structures used in **exterior differential calculus**. Define the set $\wedge \mathrm{d}\mathcal{C} = \{\mathrm{d}\zeta \wedge \mathrm{d}\eta \mid \zeta, \eta \in \mathcal{C}\}$, where $\wedge$ denotes the **wedge product** with the standard properties

$$\mathrm{d}\zeta \wedge \mathrm{d}\eta = -\mathrm{d}\eta \wedge \mathrm{d}\zeta \quad \text{and} \quad \mathrm{d}\zeta \wedge \mathrm{d}\zeta = 0$$

for $\zeta, \eta \in \mathcal{C}$.

Introduce the space $\mathcal{E}^2 = \mathrm{span}_{\mathcal{K}} \wedge \mathrm{d}\mathcal{C}$ with elements being **two-forms**. The operator $\mathrm{d} : \mathcal{E} \to \mathcal{E}^2$, called **exterior derivative** operator, is defined for $\omega = \sum_{\ell=1}^{k} \alpha_\ell(\zeta_1, \ldots, \zeta_k)\mathrm{d}\zeta_\ell \in \mathcal{E}$, where $\zeta_1, \ldots, \zeta_k \in \mathcal{C}$, by the rule

$$\mathrm{d}\omega := \sum_{\ell,\ell'} \frac{\partial \alpha_\ell}{\partial \zeta_{\ell'}} \mathrm{d}\zeta_\ell \wedge \mathrm{d}\zeta_{\ell'}.$$

# Differential forms

**Example:** Let $\omega = \mathrm{d}x_1 - \frac{x_1}{x_2}\mathrm{d}x_2$, then

$$\mathrm{d}\omega = \mathrm{d}[\mathrm{d}x_1 - \frac{x_1}{x_2}\mathrm{d}x_2] = \underbrace{\mathrm{d}[\mathrm{d}x_1]}_{=0} - \mathrm{d}\left[\frac{x_1}{x_2}\mathrm{d}x_2\right]$$
$$= -\frac{\partial}{\partial x_1}\left(\frac{x_1}{x_2}\right)\mathrm{d}x_1 \wedge \mathrm{d}x_2 - \frac{\partial}{\partial x_2}\left(\frac{x_1}{x_2}\right)\mathrm{d}x_2 \wedge \mathrm{d}x_2$$
$$\overset{\mathrm{d}x_2 \wedge \mathrm{d}x_2 = 0}{=} -\frac{1}{x_2}\mathrm{d}x_1 \wedge \mathrm{d}x_2$$

**Remark:** The notion of two-form can be generalized to the $s$-form and wedge product is defined for arbitrary $s$-forms.

# Differential forms

## Closed and exact forms

**Definition**

A one-form $\omega \in \mathcal{E}$ is **closed**, if $\mathrm{d}\omega = 0$.

**Definition**

A one-form $\omega \in \mathcal{E}$ is **exact**, if $\omega = \mathrm{d}\zeta$ for some $\zeta \in \mathcal{K}$.

**Proposition**

*Any exact one-form is closed.*

# Differential forms

## Poincaré's Lemma

**Lemma (Poincaré's Lemma)**

*Let $\omega$ be a closed one-form in $\mathcal{E}$. Then there exists $\varphi \in \mathcal{K}$ such that locally $\omega = \mathrm{d}\varphi$.*

**Example:** Consider a closed one-form

$$\omega = \frac{x_2}{x_1^2 + x_2^2}\mathrm{d}x_1 - \frac{x_1}{x_1^2 + x_2^2}\mathrm{d}x_2.$$

Locally around points

- $(x_1, x_2)$ such that $x_2 \neq 0$, we get $\omega = \mathrm{d}[\arctan(x_1/x_2)]$;
- $(x_1, x_2)$ such that $x_1 \neq 0$ and $x_2 = 0$, we get $\omega = \mathrm{d}[\arctan(-x_2/x_1)]$.

However, there is no function $\varphi$ such that $\omega = \mathrm{d}\varphi$ globally.

# Frobenius theorem

**Definition**

A subspace $\Omega \subset \mathcal{E}$ is closed or integrable, if $\Omega$ has a basis which consists only of closed forms.

**Theorem**

*Let* $\Omega = \operatorname{span}_{\mathcal{K}}\{\omega_1, \ldots, \omega_\kappa\}$*. The subspace* $\Omega$ *is integrable if and only if*

$$\mathrm{d}\omega_i \wedge \omega_1 \wedge \cdots \wedge \omega_\kappa = 0$$

*for all* $i = 1, \ldots, \kappa$*.*

# Frobenius theorem

**Example:** Consider the one-form $\omega = \mathrm{d}x_1 + x_1\mathrm{d}x_2$. To verify whether $\omega$ is closed or not we need to find the exterior derivative as

$$\mathrm{d}\omega = \mathrm{d}[\mathrm{d}x_1 + x_1\mathrm{d}x_2] = \underbrace{\mathrm{d}[\mathrm{d}x_1]}_{=0} + \mathrm{d}[x_1\mathrm{d}x_2]$$
$$= \frac{\partial x_1}{\partial x_1}\mathrm{d}x_1 \wedge \mathrm{d}x_2 + \frac{\partial x_1}{\partial x_2}\mathrm{d}x_2 \wedge \mathrm{d}x_2 = \mathrm{d}x_1 \wedge \mathrm{d}x_2.$$

Therefore, $\omega$ is not closed since $\mathrm{d}\omega \neq 0$.

However, the vector space $\mathrm{span}_{\mathcal{K}}\{\omega\}$ is integrable since

$$\mathrm{d}\omega \wedge \omega = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge (\mathrm{d}x_1 + x_1\mathrm{d}x_2)$$
$$= \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_1 + x_1\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_2 \overset{\mathrm{d}x_i \wedge \mathrm{d}x_i = 0}{=} 0.$$

Finally, if we choose the integrating factor $\alpha = 1/x_1$, then $\omega$ becomes integrable and $F = \ln|x_1| + x_2$.

# Sequence $\mathcal{H}_k$

A sequence of subspaces
$\mathcal{H}_0 \supset \cdots \supset \mathcal{H}_{k^*} \supset \mathcal{H}_{k^*+1} = \mathcal{H}_{k^*+2} = \cdots =: \mathcal{H}_\infty$ of $\mathcal{E}$ is defined by

$$\begin{aligned} \mathcal{H}_0 &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_1, \ldots, \mathrm{d}x_n, \mathrm{d}u\}, \\ \mathcal{H}_k &= \{\omega \in \mathcal{H}_{k-1} \mid \dot{\omega} \in \mathcal{H}_{k-1}\}, \quad k \geq 1, \end{aligned}$$

or

$$\begin{aligned} \mathcal{H}_1 &= \mathrm{span}_{\mathcal{K}}\left\{\mathrm{d}y, \ldots, \mathrm{d}y^{(n-1)}, \mathrm{d}u, \ldots, \mathrm{d}u^{(s)}\right\}, \\ \mathcal{H}_{k+1} &= \{\omega \in \mathcal{H}_k \mid \dot{\omega} \in \mathcal{H}_k\}, \quad k \geq 1. \end{aligned}$$

Sequence $\mathcal{H}_k$ plays an important role in the analysis of the structural properties of nonlinear systems.

# Polynomial framework
## Skew polynomial rings

**Definition**

A **skew polynomial ring** $\mathcal{A}[\partial;\alpha,\beta]$ is a noncommutative polynomial ring in $\partial$ with coefficients in $\mathcal{A}$ satisfying

$$\forall a \in \mathcal{A}, \quad \partial a = \alpha(a)\partial + \beta(a).$$

Each polynomial $\pi \in \mathcal{A}[\partial;\alpha,\beta]$ can be uniquely written in the form

$$\pi = \sum_{\ell=0}^{k} \pi_\ell \partial^{k-\ell}, \quad \pi_\ell \in \mathcal{A}.$$

If $\pi_0 \not\equiv 0$, then $k$ is called the degree of $\pi$, denoted by $\deg(\pi)$.

# Polynomial framework

## Skew polynomial rings: special cases

Several special cases:

- **Ring of differential operators:** $\mathcal{A}[\partial; \mathrm{id}, \frac{\mathrm{d}}{\mathrm{d}t}]$.
- Ring of shift operators: $\mathcal{A}[\partial; \sigma, 0]$, $\mathcal{A}[\partial; \delta, 0]$.
- Ring of difference operators: $\mathcal{A}[\partial; \tau, \tau - \mathrm{id}]$ with $\tau a(x) = a(x+1)$.

**Definition**

The skew polynomial ring, induced by $(\mathcal{K}, \mathrm{d}/\mathrm{d}t)$, is the ring $\mathcal{K}[\partial; \mathrm{id}_{\mathcal{K}}, \mathrm{d}/\mathrm{d}t] := \mathcal{K}[\partial; \mathrm{d}/\mathrm{d}t]$ of polynomials with usual addition and multiplication satisfying, for any $\varsigma \in \mathcal{K} \subset \mathcal{K}[\partial; \mathrm{d}/\mathrm{d}t]$, the commutation rule

$$\partial\varsigma := \varsigma\partial + \dot{\varsigma}.$$

# Polynomial framework

## Commutation rule: examples

**Example 1:** Consider multiplication of two polynomials $p(\partial) = \partial^2 + 1$ and $q(\partial) = y\partial - 1$

$$\begin{aligned} p(\partial)q(\partial) &= (\partial^2 + 1)(y\partial - 1) = \partial^2 y\partial - \partial^2 + y\partial - 1 \\ &= \partial(y\partial^2 + \dot{y}\partial) - \partial^2 + y\partial - 1 \\ &= y\partial^3 + \dot{y}\partial^2 + \dot{y}\partial^2 + \ddot{y}\partial - \partial^2 + y\partial - 1 \\ &= y\partial^3 + (2\dot{y} - 1)\partial^2 + (\ddot{y} + y)\partial - 1 \end{aligned}$$

**Example 2:**

$$\begin{aligned} &\partial \cdot (y + u + 1) = y\partial + \dot{y} + u\partial + \dot{u}, \\ &(y + u + 1) \cdot \partial = y\partial + u\partial + \partial. \end{aligned}$$

# Polynomial framework

## Polynomial system description

- Recall that $\mathcal{K}[\partial; \mathrm{d}/\mathrm{d}t]$ is the skew polynomial ring, where $\partial$ is a polynomial indeterminate. Multiplication in $\mathcal{K}[\partial; \mathrm{d}/\mathrm{d}t]$ is defined by the commutation rule $\partial\varsigma = \varsigma\partial + \dot{\varsigma}$, $\alpha \in \mathcal{K}$.

- Polynomial system description

$$y^{(n)} = \phi\left(y, \dot{y}, \ldots, y^{(n-1)}, u, \dot{u}, \ldots, u^{(s)}\right)$$

$$\partial^i \mathrm{d}y := \mathrm{d}y^{(i)}$$
$$\partial^j \mathrm{d}u := \mathrm{d}u^{(j)}$$

$$\left[\partial^n - \sum_{i=0}^{n-1} p_i \partial^i\right] \mathrm{d}y - \sum_{j=0}^{s} q_j \partial^j \mathrm{d}u = 0$$

$$p_i = \frac{\partial \phi}{\partial y^{(i)}} \qquad q_j = \frac{\partial \phi}{\partial u^{(j)}}$$

$$p(\partial)\mathrm{d}y + q(\partial)\mathrm{d}u = 0$$

# Polynomial framework

## Polynomial system description: Example

Consider the nonlinear system

$$\ddot{y} = \dot{u}y + u^2\dot{y}.$$

Define $\phi := \dot{u}y + u^2\dot{y}$ and differentiate it with respect to $y, \dot{y}, u$ and $\dot{u}$

$$p_0 = \frac{\partial \phi}{\partial y} = \dot{u}, \qquad p_1 = \frac{\partial \phi}{\partial \dot{y}} = u^2,$$

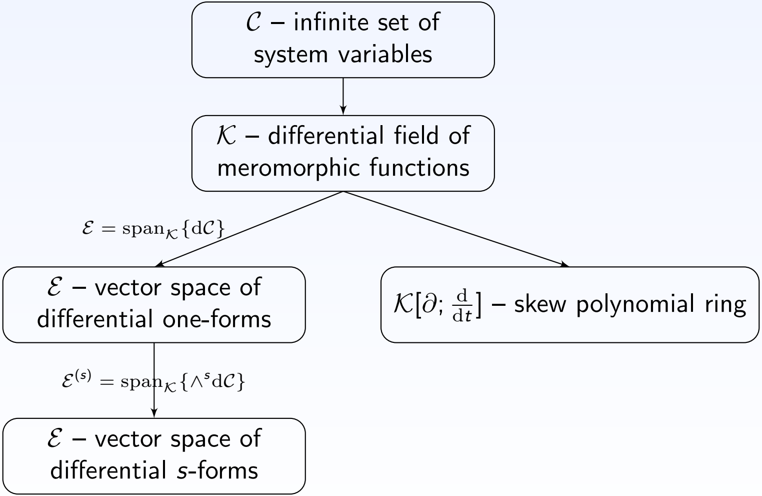
$$q_0 = \frac{\partial \phi}{\partial u} = 2u\dot{y}, \qquad q_1 = \frac{\partial \phi}{\partial \dot{u}} = y.$$

Using relations $\partial^i \mathrm{d}y := \mathrm{d}y^{(i)}$ and $\partial^j \mathrm{d}u := \mathrm{d}u^{(j)}$, we get

$$(\partial^2 - u\partial - \dot{u})\mathrm{d}y - (y\partial + 2u\dot{y})\mathrm{d}u = 0.$$

# Algebraic and polynomial formalism: Summary

## Actual picture



# Problem statement

$$y^{(n)} = \phi\left(y, \dot{y}, \ldots, y^{(n-1)}, u, \dot{u}, \ldots, u^{(s)}\right)$$

$\downarrow$ ?

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x) \end{aligned}$$

**Goal:**

Find, if possible, the state coordinates $x(t) \in \mathbb{R}^n$ such that in these coordinates the system takes the minimal state-space form.

**Definition**

The state-space description is said to be realization of the i/o equation if both equations have the same solution sets $\{(u(t), y(t)), t \geq 0\}$.

# State of the Art

- Some of the existing results are based:
  - on the sequence of distributions of vector fields

    A. J. van der Shaft, 1987
  - on the iterative Lie brackets of the vector fields

    E. Delaleau and W. Respondek, 1995
  - on the sequence of the subspaces of differential one-forms

    G. Conte, C. H. Moog, and A. M. Perdon, 2007
  - on polynomial framework

    Ü. Kotta, M. Tõnso, and J. Belikov, 2009-...

- Polynomial approach:
  - System is described by two polynomials from the skew polynomial ring
  - Solution in terms of polynomials $\Rightarrow$ explicit formulas
  - More transparent and simple $\Rightarrow$ easy to implement in symbolic software
  - Similar to the linear case $\Rightarrow$ easier to understand

# Realizability

Recall that the sequence of subspaces $\{\mathcal{H}_k\}_{k=1}^{\infty}$ of $\mathcal{E}$ is defined as

$$\mathcal{H}_1 = \operatorname{span}_{\mathcal{K}}\left\{\mathrm{d}y, \ldots, \mathrm{d}y^{(n-1)}, \mathrm{d}u, \ldots, \mathrm{d}u^{(s)}\right\},$$
$$\mathcal{H}_{k+1} = \{\omega \in \mathcal{H}_k \mid \dot{\omega} \in \mathcal{H}_k\}, \quad k \geq 1.$$

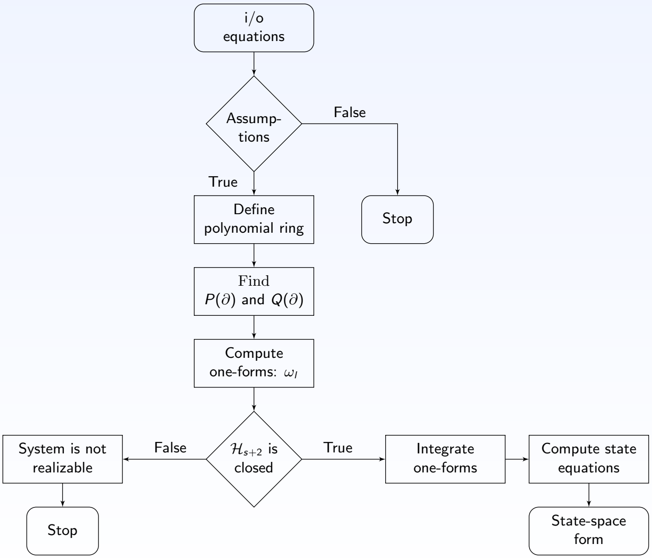
## Theorem

*The nonlinear i/o equation has an observable state-space realization if and only if the subspace $\mathcal{H}_{s+2}$ is integrable.*

## Corollary

*The state coordinates can be obtained by integrating the exact basis vectors of $\mathcal{H}_{s+2}$.*

# Realization algorithm: general idea



# Computation of $\mathcal{H}_{s+2}$: polynomial method

Subspace $\mathcal{H}_{s+2}$ can be calculated as

$$\mathcal{H}_{s+2} = \mathrm{span}_{\mathcal{K}}\{\omega_1, \ldots, \omega_n\},$$

where

$$\omega_l = \begin{bmatrix} p_l(\partial) & q_l(\partial) \end{bmatrix} \begin{bmatrix} \mathrm{d}y \\ \mathrm{d}u \end{bmatrix},$$

for $l = 1, \ldots, n$, and $p_l(\partial)$ and $q_l(\partial)$ can be recursively calculated from the equalities

$$\begin{array}{rcll} p_{l-1}(\partial) & = & \partial p_l(\partial) + \xi_l, & \deg \xi_l = 0, \\ q_{l-1}(\partial) & = & \partial q_l(\partial) + \gamma_l, & \deg \gamma_l = 0 \end{array}$$

with the initial polynomials $p_0(\partial) := p(\partial)$ and $q_0(\partial) := q(\partial)$.

# Illustrative examples

## Realizable system

Consider the nonlinear i/o system

$$\ddot{y} = \dot{u}\dot{y} + uy$$

that can be described by two polynomials

$$p(\partial) = \partial^2 - \dot{u}\partial - u \quad \text{and} \quad q(\partial) = -y\partial - u.$$

Calculate two sequences of the left quotients as: $p_1(\partial) = \partial - \dot{u}, p_2 = 1$, and $q_1(\partial) = -\dot{y}, q_2 = 0$. Then, the one-forms are

$$\begin{aligned} \omega_1 &= p_1(\partial)\mathrm{d}y + q_1(\partial)\mathrm{d}u = (\partial - \dot{u})\mathrm{d}y - \dot{y}\mathrm{d}u = \mathrm{d}\dot{y} - \dot{u}\mathrm{d}y - \dot{y}\mathrm{d}u, \\ \omega_2 &= p_2(\partial)\mathrm{d}y + q_2(\partial)\mathrm{d}u = \mathrm{d}y, \end{aligned}$$

and the subspace $\mathcal{H}_{s+2} = \mathcal{H}_3 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}\dot{y} - \dot{y}\mathrm{d}u\}$ is integrable. The choice $x_1 = y$, $x_2 = e^{-u}\dot{y}$ yields the state equations

$$\begin{aligned} \dot{x}_1 &= e^{u}x_2 \\ \dot{x}_2 &= e^{-u}ux_1 \\ y &= x_1. \end{aligned}$$

# Illustrative examples (cont.)

## Non-realizable system

Consider the "ball and beam" system

$$\ddot{y} = \frac{mR^2}{J + mR^2}\left(y\dot{u}^2 - g\sin(u)\right), \tag{1}$$

where $J, R, m, g$ are some physical parameters. The i/o equation can be described in polynomial form as

$$p(\partial) = \partial^2 - \frac{mR^2\dot{u}^2}{J + mR^2} \quad \text{and} \quad q(\partial) = -\frac{2mR^2 y\dot{u}}{J + mR^2}\partial + \frac{gmR^2\cos(u)}{J + mR^2}.$$

Compute the left quotients as: $p_1(\partial) = \partial, p_2(\partial) = 1$ and $q_1(\partial) = -\frac{2mR^2}{J+mR^2}y\dot{u}, q_2(\partial) = 0$. Then, we get $\mathcal{H}_3 = \mathrm{span}_{\mathcal{K}}\{\omega_1, \omega_2\} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}\dot{y} - \frac{2mR^2}{J+mR^2}y\dot{u}\mathrm{d}u\}$, which by the Frobenius theorem is not closed, since

$$\mathrm{d}\omega_2 \wedge \omega_1 \wedge \omega_2 = \frac{2mR^2}{J + mR^2}y\dot{u}\mathrm{d}u \wedge \mathrm{d}\dot{u} \wedge \mathrm{d}y \wedge \mathrm{d}\dot{y} \neq 0.$$

Therefore, the i/o equation does not admit the minimal state-space realization.

# Realization: open problems

**Remark:**

- The realizability conditions are constructive and can be checked using $\mathcal{H}_{s+2}$.
- To find the state coordinates, one has to integrate the differential one-forms. The integration of (integrable in principle) differential one-forms is known to be a difficult task, in general.
- Theorem (realizability) does not define explicitly the class of i/o equations that have state-space form.

Therefore, the alternative way to tackle the realization problem is to **single out** the realizable structures for low-order i/o equations as well as to understand what can happen in case of arbitrary order, suggesting some **subclasses** of general order.

# Realization: open problems

## Second-order system

Consider the second-order i/o equation

$$\ddot{y} = \phi(y, \dot{y}, u, \dot{u})$$

that can be described by two polynomials

$$p(\partial) = \partial^2 - \frac{\partial \phi}{\partial \dot{y}}\partial - \frac{\partial \phi}{\partial y}$$

and

$$q(\partial) = -\frac{\partial \phi}{\partial \dot{u}}\partial - \frac{\partial \phi}{\partial u}.$$

# Realization: open problems

## Subspace $\mathcal{H}_{s+2}$

Since $s = 1$, we have to check the integrability of the subspace $\mathcal{H}_3 = \mathrm{span}_{\mathcal{K}}\{\omega_1, \omega_2\}$, where

$$\omega_1 = p_1(\partial)\mathrm{d}y + q_1(\partial)\mathrm{d}u = \mathrm{d}\dot{y} - \frac{\partial \phi}{\partial \dot{u}}\mathrm{d}u,$$
$$\omega_2 = p_2(\partial)\mathrm{d}y + q_2(\partial)\mathrm{d}u = \mathrm{d}y.$$

The integrability can be checked using the Frobenius theorem, i.e. to check

$$\mathrm{d}\omega_i \wedge \omega_1 \wedge \cdots \wedge \omega_\kappa = 0$$

for all $i = 1, \ldots, \kappa$.

The first condition $\mathrm{d}\omega_2 \wedge \omega_1 \wedge \omega_2 = 0$ is trivially satisfied.

# Realization: open problems

The second condition $\mathrm{d}\omega_1 \wedge \omega_1 \wedge \omega_2 = 0$ can be represented as

$$\mathrm{d}\left[\mathrm{d}\dot{y} - \frac{\partial\phi}{\partial\dot{u}}\mathrm{d}u\right] \wedge \left[\mathrm{d}\dot{y} - \frac{\partial\phi}{\partial\dot{u}}\mathrm{d}u\right] \wedge \mathrm{d}y = 0$$

or

$$\left[-\frac{\partial^2\phi}{\partial\dot{u}\partial y}\mathrm{d}y \wedge \mathrm{d}u - \frac{\partial^2\phi}{\partial\dot{u}\partial\dot{y}}\mathrm{d}\dot{y} \wedge \mathrm{d}u - \frac{\partial^2\phi}{\partial\dot{u}\partial u}\mathrm{d}u \wedge \mathrm{d}u \right.$$
$$\left. - \frac{\partial^2\phi}{\partial\dot{u}\partial\dot{u}}\mathrm{d}\dot{u} \wedge \mathrm{d}u\right] \wedge \left[\mathrm{d}\dot{y} - \frac{\partial\phi}{\partial\dot{u}}\mathrm{d}u\right] \wedge \mathrm{d}y = 0.$$

# Realization: open problems

Using the basic properties of the exterior product $\mathrm{d}\zeta \wedge \mathrm{d}\zeta = 0$ and $\mathrm{d}\varepsilon \wedge \mathrm{d}\eta = -\mathrm{d}\eta \wedge \mathrm{d}\varepsilon$, the above condition can be simplified as

$$-\frac{\partial^2 \phi}{\partial \dot{u} \partial \dot{u}} \mathrm{d}u \wedge \mathrm{d}\dot{u} \wedge \mathrm{d}y \wedge \mathrm{d}\dot{y} = 0.$$

From the above equation, we get the partial differential equation

$$\frac{\partial^2 \phi}{\partial \dot{u} \partial \dot{u}} = 0.$$

The solutions of equation the obtained PDE define the complete subclass of the second-order i/o equations to be realizable in the state-space form. One particular solution is:

$$\phi = \phi_1(y, \dot{y}, u) + \phi_2(y, \dot{y}, u)\dot{u}.$$

# Realization: open problems

## Third-order system

Consider the third-order i/o equation

$$y^{(3)} = \phi(y, \dot{y}, \ddot{y}, u, \dot{u}, \ddot{u}).$$

Proceeding in the same manner as in case of the second-order i/o equation, we get the system of partial differential equations

$$\begin{cases} \phi_{\ddot{u}\ddot{u}} = 0 \\ \phi_{\ddot{u}\dot{u}} + \phi_{\ddot{u}}\phi_{\ddot{u}\ddot{y}} = 0 \\ \phi_{\dot{u}\dot{u}} - \phi_{\ddot{u}u} + 2\phi_{\ddot{u}}\phi_{\dot{u}\ddot{y}} - \phi_{\ddot{u}}\phi_{\ddot{u}\dot{y}} - (\phi_{\dot{u}} + \phi_{\ddot{u}}\phi_{\ddot{y}})\phi_{\ddot{u}\ddot{y}} + \phi_{\ddot{u}}\phi_{\ddot{u}}\phi_{\ddot{y}\ddot{y}} = 0, \end{cases}$$

where $\phi_{\alpha\beta} = \frac{\partial^2\phi}{\partial\beta\partial\alpha}$ is used to denote the partial derivative of a function.