

ISS0031 Modeling and Optimization

Practice in MATLAB: Control System Toolbox

Control System toolbox: basic functions

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In order to generate random continuous-time model of the n th order with p outputs and m inputs, the following command can be used

```
>> sys = rss(n, p, m) % random test model
```

In order to create a SS object representing the continuous-time state-space model from the known matrices A , B , C and D , the following command can be used

```
>> sys_ss = ss(A, B, C, D) % creates a SS object
```

Controllability and observability properties can be checked as

```
>> Qc = ctrb(sys) % compute controllability matrix for the model  
>> rank(Qc) % rank of the controllability matrix
```

and

```
>> Qo = obsv(sys) % compute observability matrix for the model  
>> rank(Qo) % rank of the observability matrix
```

Eigenvalues of the system can be calculated by means of the following function

```
>> eig(A) % eigenvalues
```

In order to find transfer function description of the system, the following command can be used

```
>> tf(sys) % convert the model to transfer function form
```

LTI Viewer: allows to perform a brief analysis of the system

```
>> ltiview(sys) % open an LTI Viewer containing the step response
```

Pole placement

Full state feedback or pole placement, is a method employed in feedback control system theory to place the closed-loop poles of a plant in pre-determined locations in the s -plane. Placing poles is desirable because the location of the poles corresponds directly to the eigenvalues of the system, which control the characteristics of the response of the system. The system must be considered controllable in order to implement this method.

Let a system is given in the state-space form

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}\tag{1}$$

The poles of the system are the roots of the characteristic equation given by

$$|sI - A| = 0.$$

We shall choose the control signal to be

$$u = -Kx. \quad (2)$$

This means that the control signal u is determined by an instantaneous state. Such a scheme is called state feedback. The $1 \times n$ matrix K is called the state feedback gain matrix. We assume that all state variables are available for feedback. The latter comes from the complete state controllability property of the system.

Substituting equation (2) into equation (1) gives

$$\begin{aligned} \dot{x} &= (A - BK)x \\ y &= (C - DK)x \end{aligned}$$

It obvious that the stability and transient response characteristics are determined by the eigenvalues of matrix $A - BK$. If matrix K is chosen properly, the matrix $A - BK$ can be made an asymptotically stable matrix. The eigenvalues of matrix $A - BK$ are called the regulator poles, which have to be placed in the left-half s plane. The latter can be done by calculating roots of the characteristic equation, $\det[sI - (A - BK)]$ and comparing the terms of this equation with those of the desired characteristic equation.

Example 1: Given state-space model of the inverted pendulum on a cart

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

in which $x = [\theta \ \dot{\theta} \ d \ \dot{d}]^T$, θ [rad] is angle of pendulum, d [m] is position of a cart with respect to the origin, u [N] is force of motion (model input),

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{M+m}{M \cdot l}g & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{m}{M}g & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \frac{1}{M} \end{bmatrix}, \quad C = I_{4 \times 4}, \quad D = 0_{4 \times 4}$$

and parameters are defined as: mass of cart $M = 5$ [kg], mass of pendulum $m = 0.5$ [kg], distance to pendulum center of mass $l = 0.5$ [m], acceleration of gravity $g = 9.81$ [m/s²].

The matrix K can be calculated using MATLAB function `place` as follows

```
>> g = 9.8;
>> M = 2;
>> m = 0.2;
>> l = 0.6;
```

```

% matrices of the model
>> A = [0 1 0 0; g*(M + m)/(M*1) 0 0 0; 0 0 0 1; -m/M*g 0 0 0]
>> B = [0; -1/(M*1); 0; 1/M]
>> C = eye(4) % unit matrix
>> D = zeros(4,1) % zero matrix

>> sys = ss(A, B, C, D) % create state-space model

% controllability
>> Qc = ctrb(sys)
>> rank(Qc)

>> eig(sys) % eigenvalues of the system

% Synthesis of the closed-loop system
>> s1 = -1; s2 = -2; s3 = -3; s4 = -4; % desired poles
>> S = [s1 s2 s3 s4]

>> K = place(sys.a, sys.b, S) % calculate values for the matrix K

```

Solving quadratic optimal regulator problems with MATLAB

The material presented in this section is based on Ogata's book (Section 10-8). An advantage of the quadratic optimal control method over the pole-placement method is that the former provides a systematic way of computing the state feedback control gain matrix.

In MATLAB, the command

```
lqr(A, B, Q, R)
```

solves the continuous-time, linear, quadratic regulator problem and the associated Riccati equation. This command calculates the optimal feedback gain matrix K such that the feedback control law

$$u = -Kx$$

minimizes the performance index

$$J = \int_0^{\infty} (x^T Q x + u^T R u) dt$$

subject to the constraint equation

$$\dot{x} = Ax + Bu.$$

Another command

```
[K, P, E]=lqr(A, B, Q, R)
```

returns the gain matrix K , eigenvalue vector E , and matrix P , the unique positive-definite solution to the associated matrix Riccati equation

$$PA + A^T P - PBR^{-1}B^T P + Q = 0.$$

Note that if matrix $A - BK$ is a stable matrix, such a positive-definite solution P always exists. The eigenvalue vector E gives the closed-loop poles of $A - BK$. It is important to note that for certain systems matrix $A - BK$ cannot be made a stable matrix, whatever K is chosen. In such a case, there does not exist a positive-definite matrix P for the matrix Riccati equation. For such a case, the commands

```
K=lqr(A,B,Q,R)
[K,P,E]=lqr(A,B,Q,R)
```

do not give the solution.

Example 2: Consider the system described by

$$\dot{x} = Ax + Bu,$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The performance index J is given by

$$J = \int_0^{\infty} (x^T Q x + u^T R u) dt,$$

where

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = [1].$$

Assume that the following control u is used

$$u = -Kx.$$

Determine the optimal feedback gain matrix K . The optimal feedback gain matrix K can be obtained by solving the following Riccati equation for a positive-definite matrix P

$$A^T P + PA - PBR^{-1}B^T P + Q = 0.$$

The result is

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Substituting this P matrix into the following equation gives the optimal K matrix

$$K = R^{-1}B^T P = [1] \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = [1 \quad 1].$$

Thus, the optimal control signal is given by

$$u = -Kx = -x_1 - x_2.$$

The same problem can be solved by MATLAB

```
A = [0 1;0 -1];
B = [0;1];
Q = [1 0; 0 1];
R = [1];
K = lqr(A,B,Q,R)
K =
1.0000 1.0000
```