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TALLINN UNIVERSITY OF  
TECHNOLOGY

ISS0031 Modeling and Identification

**Lecture 9: Brief Introduction to the Analysis of Non-linear Systems. Phase Plane analysis. Linearization.**

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# Recall: State Space Representation of Dynamic Systems

Suppose that a system has  $p$  inputs  $u_i(t)$ ,  $i = 1, 2, \dots, p$ , and  $q$  outputs  $y_k(t)$ ,  $k = 1, 2, \dots, q$ , and there are  $n$  system states that make up a state variable vector  $x = [x_1 \ x_2 \ \dots \ x_n]^T$ . The state space expression of general dynamic systems can be written as

$$\begin{cases} \dot{x}_i = f_i(x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_p), & i = 1, 2, \dots, n, \\ y_k = g_k(x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_p), & k = 1, 2, \dots, q, \end{cases} \quad (1)$$

where  $f_i(\cdot)$  and  $g_k(\cdot)$  can be any nonlinear functions. For linear, time-invariant systems the state space expression is

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du, \end{cases} \quad (2)$$

where  $u = [u_1 \ u_2 \ \dots \ u_n]^T$  and  $y = [y_1 \ y_2 \ \dots \ y_q]^T$  are the input and output vectors, respectively. The matrices  $A$ ,  $B$ ,  $C$ , and  $D$  are compatible matrices with sizes  $n \times n$ ,  $n \times p$ ,  $q \times n$ , and  $q \times p$ , respectively.



# Examples of Nonlinear Phenomena

- **Finite escape time.** The state of an unstable nonlinear system can go to infinity in finite time, whereas the state of an unstable linear system goes to infinity as time approaches infinity.
- **Multiple isolated equilibrium points.** A nonlinear system may have more than one isolated equilibrium point. The state may converge to one of several steady-state operating points. A linear system, on the other hand, can have only one isolated equilibrium point. Therefore, it has a single steady-state operating point that attracts the state of the system without regard to the initial state.
- **Limit cycles.** A stable oscillation of fixed amplitude and frequency irrespective on the initial state should be produced by a nonlinear system, since for a linear system to oscillate a nonrobust condition must be fulfilled—it will be very difficult to maintain stable oscillation in the presence of perturbations (e.g., disturbances).



# Examples of Nonlinear Phenomena (continued)

- **(Sub)harmonic or almost periodic oscillations.** Under a periodic input signal a nonlinear system may not produce an output of the same frequency; moreover, it may produce an almost periodic signal. On the other hand, a stable linear system would produce an output of the same frequency.
- **Chaos.** A nonlinear system may have a complicated steady-state behavior that is not equilibrium, or (almost) periodic oscillation. The system may also exhibit random behavior.
- **Multiple modes of behavior.** A nonlinear system may exhibit two or more modes of behavior. For example, an unforced system may have more than one limit cycle. A forced system with periodic excitation may exhibit (sub)harmonic or a more complicated steady-state behavior, depending on the amplitude and frequency of the input. It may even exhibit a discontinuous jump in the mode of behavior as the input excitation is smoothly changed.



# Example: Pendulum Equation

The law of motion is described by

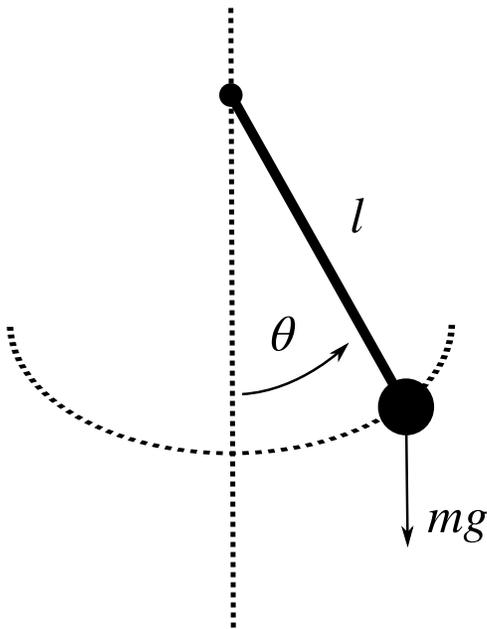
$$ml\ddot{\theta} = -mg \sin \theta - kl\dot{\theta}, \quad (3)$$

where  $m$  is the mass of the payload,  $l$  is the length of the rod,  $g$  is gravitational acceleration, and  $k$  is a friction coefficient. Take  $x_1 = \theta$  and  $x_2 = \dot{\theta}$ . The state model is obtained as

$$\dot{x}_1 = x_2 \quad (4)$$

$$\dot{x}_2 = -\frac{g}{l} \sin x_1 - \frac{k}{m}x_2. \quad (5)$$

Solving for zero dynamics yields equilibrium points that are located at  $x_n^0 = (n\pi, 0)$ , for  $n = 0, \pm 1, \pm 2, \dots$

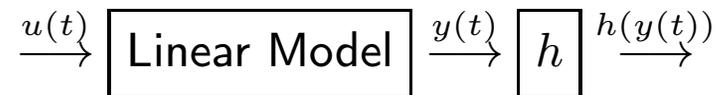


# Modeling Dynamic Systems: Dealing with Input and Output Nonlinearities

- **Hammerstein model:** (Static) input nonlinearity:



- **Wiener model:** (Static) output nonlinearity:



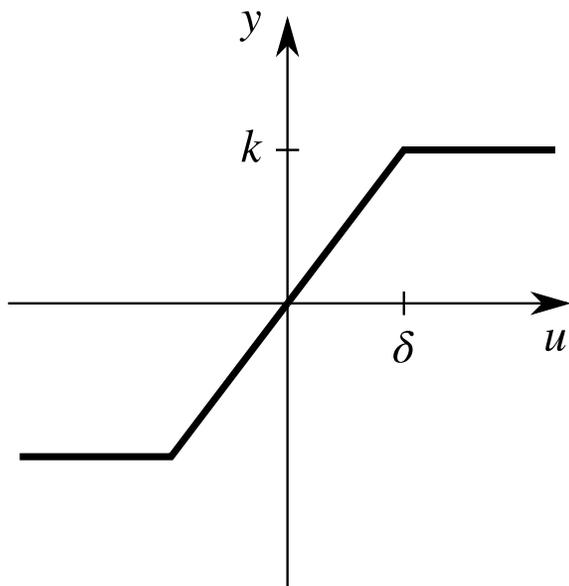
The functions  $f(\cdot)$  and  $h(\cdot)$  may be parameterized either in terms of physical parameters (e.g., saturation levels), or as (nonlinear) black-box models.

Input and output nonlinearities may also be of *dynamic* nature, i.e., exhibit varying effects that are subject to change as time passes.

The linear and nonlinear parts may be identified together, or independently, or be constructed from prior knowledge (i.e., using the *grey box* approach).



# Common Nonlinearities: Ideal Saturation



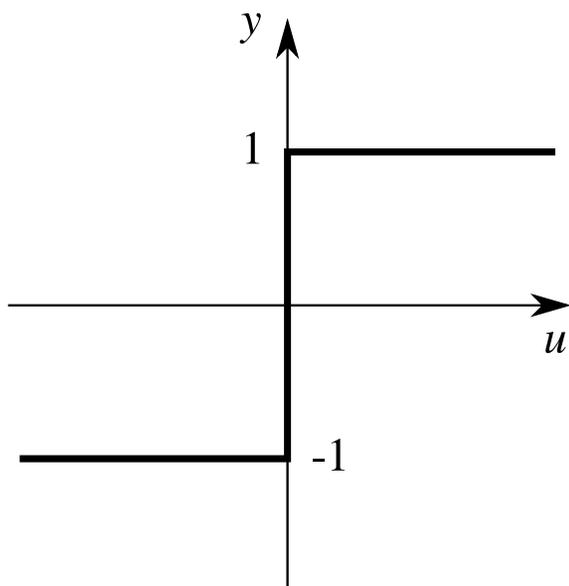
The saturation function is defined as

$$\text{sat}(u) = \begin{cases} u, & \text{if } |u| \leq 1 \\ \text{sgn}(u), & \text{if } |u| > 1. \end{cases}$$

Saturation characteristics are common in all practical amplifiers (electronic, magnetic, pneumatic, or hydraulic), motors, and actuators. This effect is also used intentionally to restrict the range of a signal.



# Common Nonlinearities: Ideal Relay



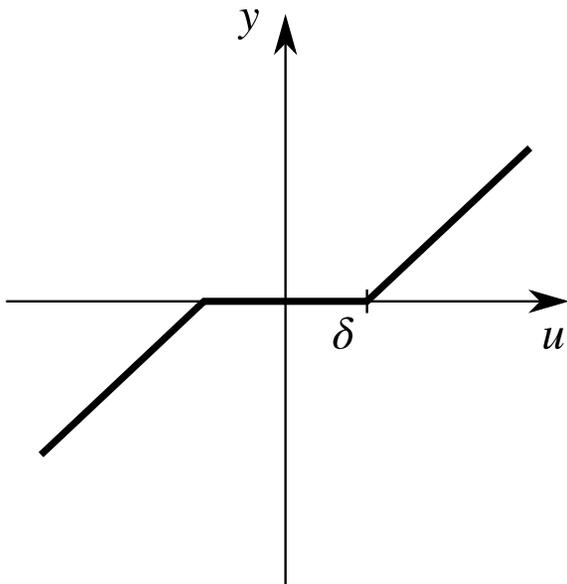
An ideal relay may be described by the signum function as

$$\text{sgn}(u) = \begin{cases} 1, & \text{if } u > 0 \\ 0, & \text{if } u = 0 \\ -1, & \text{if } u < 0. \end{cases}$$

This nonlinear characteristic can model electromechanical relays, thyristor circuits, and other switching devices. The relay characteristic is also used in various control applications (e.g., heating), where it is usual to consider *hysteresis*.



# Common Nonlinearities: Ideal Dead Zone



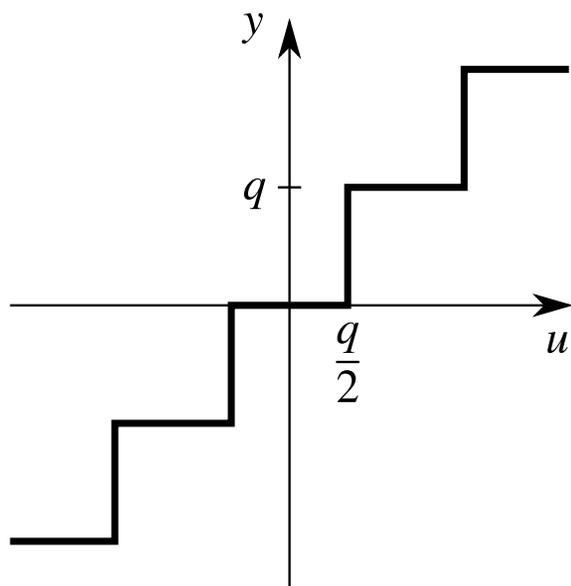
The dead-zone (also called *deadband*) characteristic may be defined by a function

$$d(u) = \begin{cases} 0, & \text{if } |u| \leq \delta \\ u, & \text{if } |u| > \delta. \end{cases}$$

This characteristic is typical of valves and some amplifiers at low input signals. The *backlash* effect known from gear interactions in a mechanical system may be approximated by the dead-zone effect.



# Common Nonlinearities: Ideal Quantization



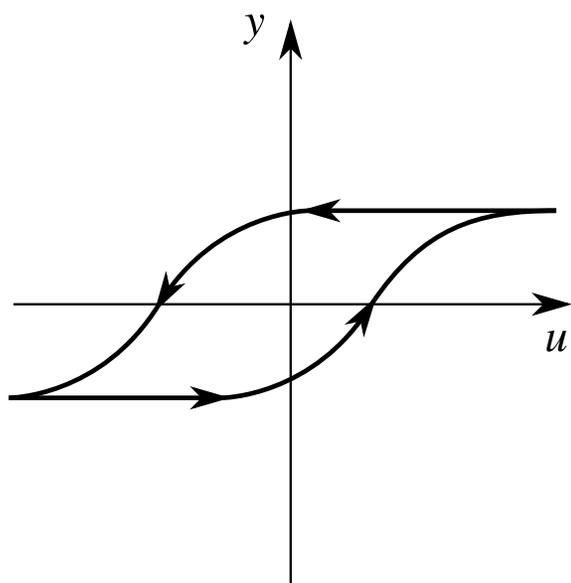
Quantization arises in real-life digital signal sampling problems. The values of the collected samples  $s_k$  will belong to a set of isolated values

$$s_k \in S \subset \mathbb{R}.$$

In case of feedback control this may lead to a limit cycle in the steady state.



# Common Nonlinearities: Hysteresis



Hysteresis is an example of a nonlinear effect with memory. Relay hysteresis usually has a rectangular shape.

In applications related to automatic tuning of controllers, different frequency points may be obtained by means of the relay experiment by changing the width of hysteresis.

Such nonlinearity is also the basis for the backlash characteristic in gears.



# Second-order Systems: Essential Definitions

A second-order autonomous system is represented by two scalar differential equations

$$\begin{cases} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2). \end{cases} \quad (6)$$

Let  $x(t) = (x_1(t), x_2(t))$  be the solution of (6) that starts at a certain initial state  $x_0 = (x_{10}, x_{20})$ . The locus in the  $(x_1-x_2)$ -plane of the solution  $x(t)$  for all  $t \geq 0$  is a curve that passes through the point  $x_0$ . This curve is called a *trajectory* or *orbit* of (6) from  $x_0$ . The  $(x_1-x_2)$ -plane is called *state plane* or *phase plane*. Using vector notation

$$\dot{x} = f(x), \quad (7)$$

where  $f(x) = (f_1(x), f_2(x))$  we consider  $f(x)$  as a *vector field* on the state plane. The family of all trajectories is called the *phase portrait* of the system (6).



# Qualitative Behavior of Linear Systems

We consider a system of the form

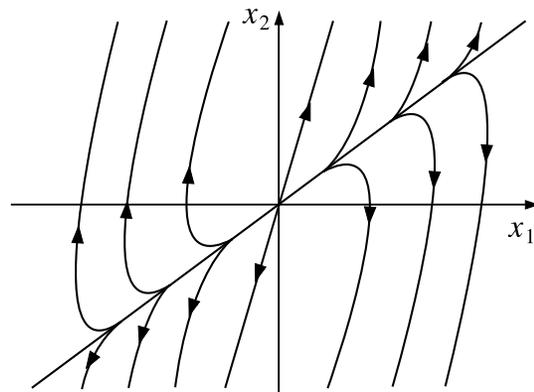
$$\dot{x} = Ax, \quad (8)$$

where  $A$  is a  $2 \times 2$  real matrix which represents a linearization of (6) near an equilibrium point. The behavior of the system depends on the eigenvalues of  $A$ .

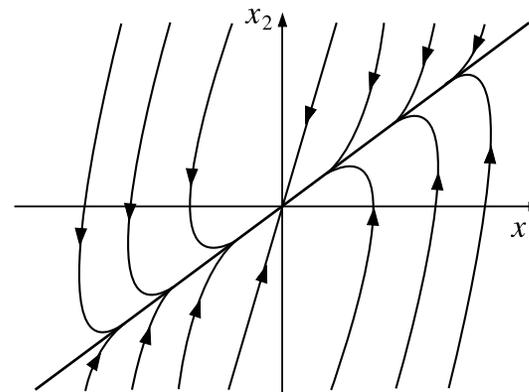
Eigenvalues of $A$	Type of equilibrium point of $\dot{x} = f(x)$
$\lambda_2 < \lambda_1 < 0$	Stable Node
$\lambda_2 > \lambda_1 > 0$	Unstable Node
$\lambda_2 < 0 < \lambda_1$	Saddle
$\alpha \pm j\beta, \alpha < 0$	Stable Focus
$\alpha \pm j\beta, \alpha > 0$	Unstable Focus
$\alpha \pm j\beta, \alpha = 0$	Center (Linearization Fails)



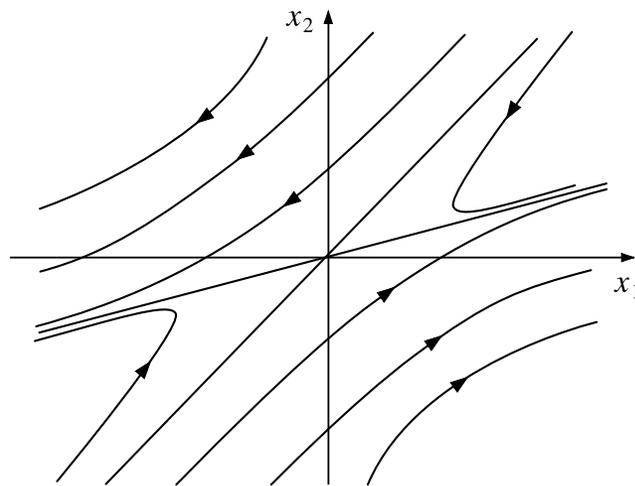
# Qualitative Behavior of Linear Systems (continued)



Unstable Node



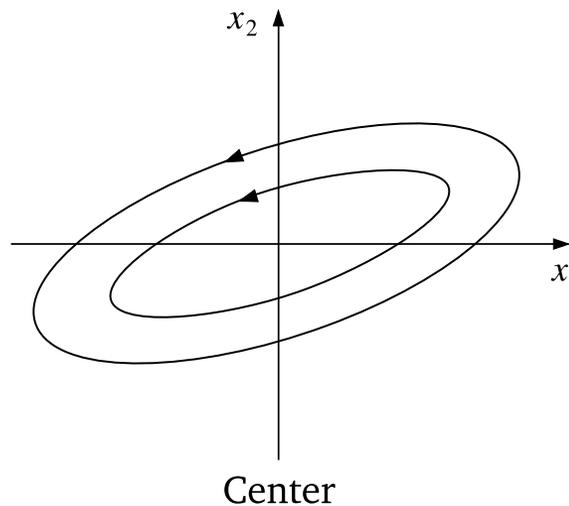
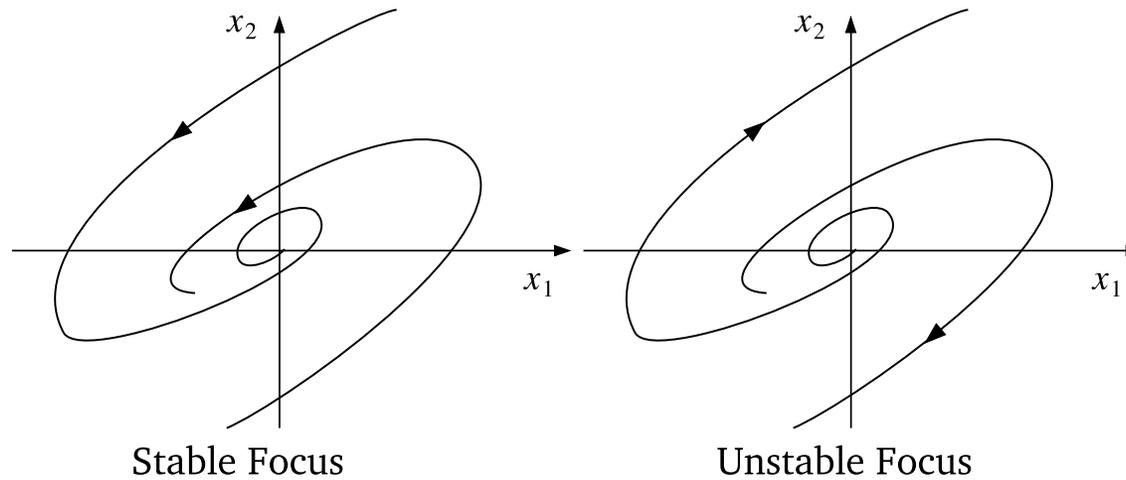
Stable Node



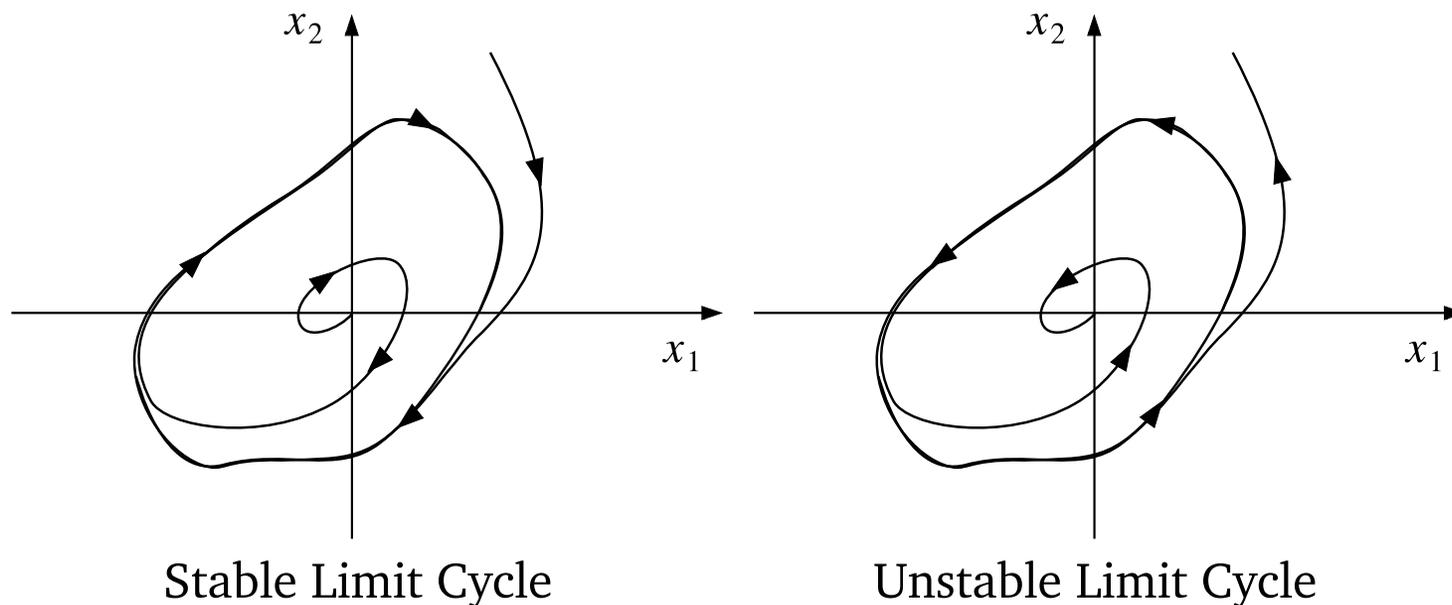
Saddle



# Qualitative Behavior of Linear Systems (continued)



# Limit Cycles



An isolated periodic orbit is called a *limit cycle*. The limit cycle is *stable* if all trajectories in the vicinity of the limit cycle ultimately tend toward the limit cycle, and *unstable* if all trajectories starting from points arbitrarily close to the limit cycle will tend away from it, as  $t \rightarrow \infty$ .



# Nonlinear Systems: Definition of Stability

**Definition 1.** The equilibrium point  $x = 0$  of an autonomous system defined by  $\dot{x} = f(x)$  is

- **stable** if for each  $\varepsilon > 0$  there is  $\delta = \delta(\varepsilon) > 0$  such that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall t \geq 0, \quad (9)$$

- **unstable** if it is not stable, i.e., condition (9) does not hold,
- **asymptotically stable** if it is stable and for some  $c > 0$  it holds

$$\|x(0)\| < c \Rightarrow \lim_{t \rightarrow \infty} x(t) = 0, \quad (10)$$

- **exponentially stable** if for some  $c > 0$ ,  $\lambda > 0$ ,  $k \geq 1$  it holds

$$\|x(0)\| < c \Rightarrow \|x(t)\| \leq k \|x(0)\| \exp(-\lambda t), \quad \forall t \geq 0. \quad (11)$$



# Linearization: Data Fitting (Identification)

As a nonlinear system may have several operating points, the approach to linearization may be the following:

1. Choose a set of  $n$  operating points of the form  $(u_k, y_k)$ ,  $k = 1, 2, \dots, n$ . The number of operating points depends on the dynamic range of the system output and/or the control problem.
2. Obtain a linear model in the vicinity of each of the operating points, e.g.:
  - (a) Perform a step experiment and use time-domain identification;
  - (b) Use the relay feedback method to collect frequency response points and use frequency-domain identification.
3. Design a switching law for the set of linear systems to form the complete nonlinear model and validate it.
4. If only control is desired, design a (linear) controller for each operating point and design a control switching law.



# Feedback Linearization: Motivation: Pendulum Example

Suppose that we want to stabilize the pendulum in (3) at an angle  $\theta = \delta$  using a torque input  $u_T = cu$ . The state equation is now

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a(\sin(x_1 + \delta) - \sin \delta) - bx_2 + cu.\end{aligned}$$

By inspection, the following choice of input

$$u = \frac{a}{c}(\sin(x_1 + \delta) - \sin \delta) + v$$

will cancel the nonlinear term  $-a(\sin(x_1 + \delta) - \sin \delta)$ . This cancellation results in a linear system with a new input

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -bx_2 + cv,\end{aligned}$$

which may be stabilized by the appropriate choice of  $v$ , e.g.,  $v = k_1x_1 + k_2x_2$ .



# Feedback Linearization: General Result

**Definition 2.** *A nonlinear system*

$$\dot{y} = f(y) + G(y)u, \quad (12)$$

where  $f : \mathbb{U} \rightarrow \mathbb{R}^n$  and  $G : \mathbb{U} \rightarrow \mathbb{R}^{n \times p}$  are sufficiently smooth on a domain  $\mathbb{U} \subset \mathbb{R}^n$ , is said to be **feedback linearizable** if there exists a diffeomorphism  $T : \mathbb{U} \rightarrow \mathbb{R}^n$  such that  $D = T(\mathbb{U})$  contains the origin and the change of variables  $x = T(y)$  transforms the system (12) into the form

$$\dot{x} = Ax + B\beta^{-1}(x)(u - \alpha(x)) \quad (13)$$

with  $(A, B)$  controllable and  $\beta(x)$  nonsingular for all  $x \in D$ .



# Where To Go From Here: Approaches to Nonlinear System Analysis

## 1. Differential Geometry:

- Application of differential and integral calculus to the study of problems in geometry.
- Reference: H. Khalil, *Nonlinear Systems*, 3rd ed., Prentice Hall, Upper Saddle River, NJ, 2002.

## 2. Differential Algebra:

- Study of various algebraic objects (differential rings, fields, and algebras) and their application to the study of differential equations.
- Reference: G. Conte, C.H. Moog, A.M. Perdon, *Algebraic Methods for Nonlinear Control Systems: Theory and Applications*, 2nd ed., Springer-Verlag, London, 2007.



# Questions?

Thank you for your attention!

