Control Lab
www.a-lab.ee

TALLINN UNIVERSITY OF TECHNOLOGY

ISS0031 Modeling and Identification

Lecture 5: Linear Systems. Transfer functions. Frequency Domain Analysis. Basic Control Design.

Aleksei Tepljakov, Ph.D.

October 3, 2017

Linear Dynamic Systems

Definition 1. *A dynamic system $y(t) = F(u(t))$ is called **linear** if for any $\alpha, \beta \in \mathbb{R}$ the* superposition *property holds:*

$$F[\alpha u_1(t) + \beta u_2(t)] = \alpha F[u_1(t)] + \beta F[u_2(t)]. \tag{1}$$

- Most (all) real-life systems are nonlinear.
- We make use of linear approximations of nonlinear systems in order to gain certain advantages in terms of application of powerful tools for analysis of linear systems.

Therefore, when dealing with real-life problems it makes sense to try to break complex systems down into an interconnected collection of smaller linear system approximations.

Laplace Transform

A function $F(s)$ of the complex variable $s = \sigma + j\omega$ is called the **Laplace transform** of the original function $f(t)$ and defined as

$$F(s) = \mathscr{L}\left[f(t)\right] = \int_0^\infty \mathrm{e}^{-st} f(t) \mathrm{d}t \tag{2}$$

The original function $f(t)$ can be recovered from the Laplace transform $F(s)$ by applying the **inverse Laplace transform**

$$f(t) = \mathscr{L}^{-1}\left[F(s)\right] = \frac{1}{j2\pi} \int_{c-j\infty}^{c+j\infty} \mathrm{e}^{st} F(s) \mathrm{d}s, \tag{3}$$

where c is greater than the real part of all the poles of $F(s)$.

Laplace Transform of a Derivative

Theorem 1 (Real Differentiation Theorem). *The Laplace transform of the derivative of a function $f(t)$ is given by*

$$\mathscr{L}\left[\frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2}\dot{f}(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0). \quad (4)$$

where $f(0), \dot{f}(0), \ldots, f^{(n-1)}(0)$ represent the values of derivatives $f(t), \mathrm{d}f(t)/\mathrm{d}t, \ldots, \mathrm{d}^{n-1}f(t)/\mathrm{d}t^{n-1}$, respectively, evaluated at $t = 0$.

For zero initial conditions $f(0) = \dot{f}(0) = \cdots = f^{(n-1)}(0) = 0$ we have

$$\mathscr{L}\left[\frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}\right] = s^n F(s).$$

Laplace Transform: Application Example: Solving Differential Equations

Task: Find the solution $x(t)$ of the differential equation

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + 3\frac{\mathrm{d}x(t)}{\mathrm{d}t} + 2x(t) = 0$$

with initial conditions $x(0) = a, \dot{x}(0) = b$.

Solution: Apply the Laplace transform:

$$\left[s^2 X(s) - sx(0) - \dot{x}(0)\right] + \left[3sX(s) - x(0)\right] + 2X(s) \; = \; 0.$$

Substitute $x(0) = a$ and $\dot{x}(0) = b$ and obtain

$$(s^2 + 3s + 2)X(s) = as + b + 3a.$$

Laplace Transform: Application Example: Solving Differential Equations (continued)

Now, solve for $X(s)$ and obtain

$$X(s) = \frac{as + b + 3a}{(s^2 + 3s + 2)} = \frac{as + b + 3a}{(s+1)(s+2)} = \frac{2a+b}{s+1} - \frac{a+b}{s+2}.$$

The inverse Laplace transform of $X(s)$ yields

$$\begin{aligned} x(t) &= \mathscr{L}^{-1}[X(s)] = \mathscr{L}^{-1}\left[\frac{2a+b}{s+1}\right] - \mathscr{L}^{-1}\left[\frac{a+b}{s+2}\right] \\ &= (2a+b)\mathrm{e}^{-t} - (a+b)\mathrm{e}^{-2t}, \quad \text{for } t \geqslant 0, \end{aligned}$$

which is the solution of the given differential equation.

Linear SISO Dynamic Systems: Differential Equations and Transfer Functions

A linear, continuous-time, single input, single output dynamic system can be expressed by a differential equation

$$a_n \frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n} + a_{n-1} \frac{\mathrm{d}^{n-1} y(t)}{\mathrm{d}t^{n-1}} + \cdots + a_1 \frac{\mathrm{d}y(t)}{\mathrm{d}t} + a_0 y(t) = \\ b_m \frac{\mathrm{d}^m u(t)}{\mathrm{d}t^m} + b_{m-1} \frac{\mathrm{d}^{m-1} u(t)}{\mathrm{d}t^{m-1}} + \cdots + b_1 \frac{\mathrm{d}u(t)}{\mathrm{d}t} + b_0 u(t), \quad (5)$$

where $a_k,\ b_k \in \mathbb{R}$. Applying the Laplace transform to (5) with zero initial conditions we obtain the **transfer function** representation of the dynamic system as the ratio of polynomials

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}. \quad (6)$$

Transfer Functions

Consider a transfer function given by

$$G(s) = \frac{B(s)}{A(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}.$$

- The polynomial $B(s)$ is called the **zero polynomial**. The roots of $B(s) = 0$ are called the **zeros** of $G(s)$.
- The polynomial $A(s)$ is called the **pole polynomial** or **characteristic polynomial** of $G(s)$. Roots of $A(s) = 0$ are called the **poles** of $G(s)$ and determine the overall behavior of the system.
- The **order** of the system is determined by the highest power n of s^n appearing in the characteristic polynomial.
- The system $G(s)$ is called **proper**, if $n \geqslant m$, and **strictly proper**, if $n > m$. A strictly proper system always satisfies $G(s) \to 0$ as $s \to \infty$. Only proper systems are realizable in practice. The value $\psi = n - m$ is called the **relative order** of the system.

Definition 2. *A system (6) is said to be* ***asymptotically stable****, when all poles* λ_i*, i.e. roots of of* $A(s) = 0$*, have negative real parts, that is*

$$\Re(\lambda_i) < 0, \quad i = 1, 2, \ldots, n. \tag{7}$$

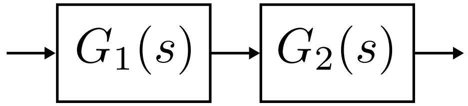
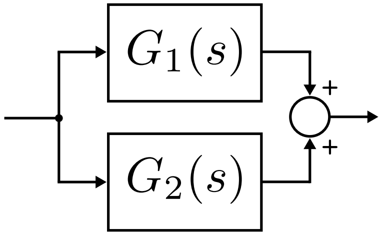
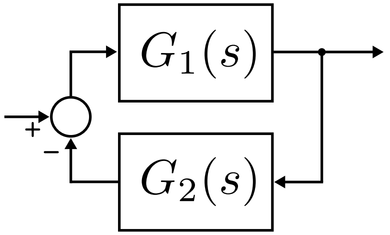
Definition 3. *The system reaction* $h(t)$ *to a unit impulse input is called the* ***impulse response*** *of the system:*

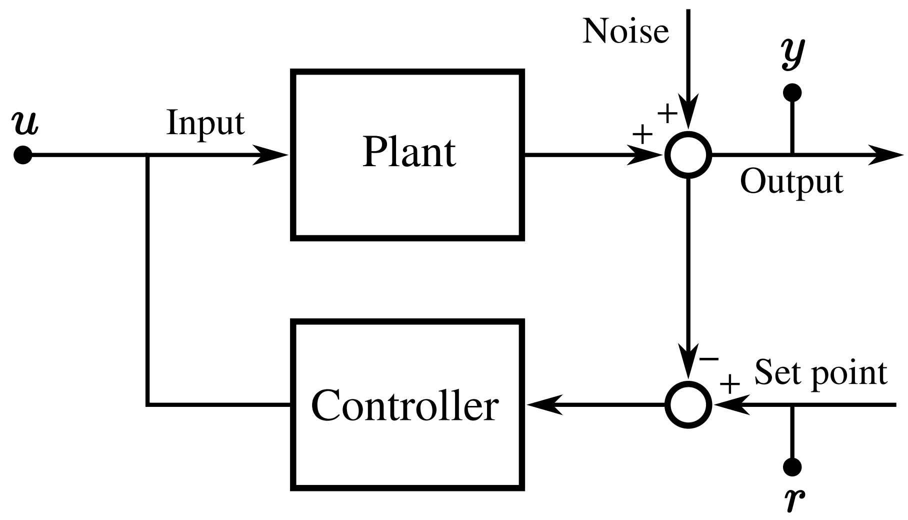
$$h(t) \overset{\mathscr{L}}{\longleftrightarrow} H(s) = \frac{B(s)}{A(s)}. \tag{8}$$

Definition 4. *The system reaction* $g(t)$ *to a unit step input is called the* ***step response*** *of the system:*

$$g(t) \overset{\mathscr{L}}{\longleftrightarrow} \frac{1}{s}H(s) = \frac{1}{s}\frac{B(s)}{A(s)}. \tag{9}$$

Transfer Functions: System Interconnections

$G_1(s) \cdot G_2(s)$ (series connection)	
$G_1(s) + G_2(s)$ (parallel connection)	
$\dfrac{G_1(s)}{1 + G_1(s)G_2(s)}$ (negative feedback)	



In SISO control, usage of PID (Proportional-Integral-Derivative) controllers is commonplace. The parallel form of the conventional PID controller is given by

$$C(s) = K_p + K_i s^{-1} + K_d s. \tag{10}$$

Process Models

- First order systems possess the most common behaviour encountered in practice;
- First order processes are characterized by
  - Their capacity to store material, momentum and energy;
  - The resistance associated with the flow of mass, momentum, or energy in reaching their capacity.
- Common first-order process models take the system transport delay L into account;
- Control design for first-order processes is reasonably well studied. In particular, numerous tuning rules for PID controllers are available. See, e.g., A. O'Dwyer, *Handbook of PI and PID Controller Tunning Rules*, 3rd ed. Imperial College Press, 2009.

First Order Plus Dead Time (FOPDT) Process Model

Transfer function:

$$G(s) = \frac{K}{Ts+1}\mathrm{e}^{-Ls}, \tag{11}$$

where K is the **static gain**, T is the **time constant** (the time it takes for the dynamic system to reach 63.2% of its total change without regard to the transport delay), and L is the **transport delay**.

Impulse response:

$$h(t) = \frac{K}{T} \cdot \mathrm{e}^{-\frac{(t-L)}{T}} \cdot \theta(t-L), \tag{12}$$

where $\theta(\cdot)$ is the unit step (Heaviside) function.

Step response:

$$g(t) = K \cdot (1 - \mathrm{e}^{-\frac{(t-L)}{T}}) \cdot \theta(t-L). \tag{13}$$

Integrator Plus Dead Time (IPDT) Process Model

Transfer function:

$$G(s) = \frac{K}{s}\mathrm{e}^{-Ls}, \tag{14}$$

where K is the gain, and L is the transport delay.

Impulse response:

$$h(t) = K \cdot \theta(t - L). \tag{15}$$

Step response:

$$g(t) = K \cdot (t - L) \cdot \theta(t - L). \tag{16}$$

First Order Integrator Plus Dead Time (FOIPDT) Process Model

Transfer function:

$$G(s) = \frac{K}{s(Ts+1)}\mathrm{e}^{-Ls}, \tag{17}$$

where K is the gain, T is the time constant, and L is the transport delay.

Impulse response:

$$h(t) = K \cdot (1 - \mathrm{e}^{-\frac{(t-L)}{T}}) \cdot \theta(t-L). \tag{18}$$

Step response:

$$g(t) = K(T\mathrm{e}^{-\frac{(t-L)}{T}} - L + t - T)\theta(t-L). \tag{19}$$

Frequency Domain Analysis of SISO Systems†

Recall that for a function $f(x)$ the Fourier series is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx). \tag{20}$$

This means that any signal $u(t)$ can be represented by a Fourier series, i.e., any signal can be seen as an infinite sum of weighted harmonic functions. Now consider an input signal $u(t) = \sin(\omega t)$ passed through a transfer function $H(s)$ such that

$$\xrightarrow{u(t)} \boxed{H(s)} \xrightarrow{y(t)}$$

The steady-state output will be $y = A\sin(\omega t + \phi)$.

† — more detail at `http://a-lab.ee/edu/ajs/freq/`

Frequency Domain Analysis of SISO Systems (continued)

To construct the magnitude and phase response of a system represented by a transfer function $G(s)$ with transport delay L substitute $s = j\omega$. Then, for

$$G(j\omega) = \frac{b_m(j\omega)^m + b_{m-1}(j\omega)^{m-1} + \cdots + b_1(j\omega) + b_0}{a_n(j\omega)^n + a_{n-1}(j\omega)^{n-1} + \cdots + a_1(j\omega) + a_0} \mathrm{e}^{-L(j\omega)} \tag{21}$$

for a particular frequency ω_k

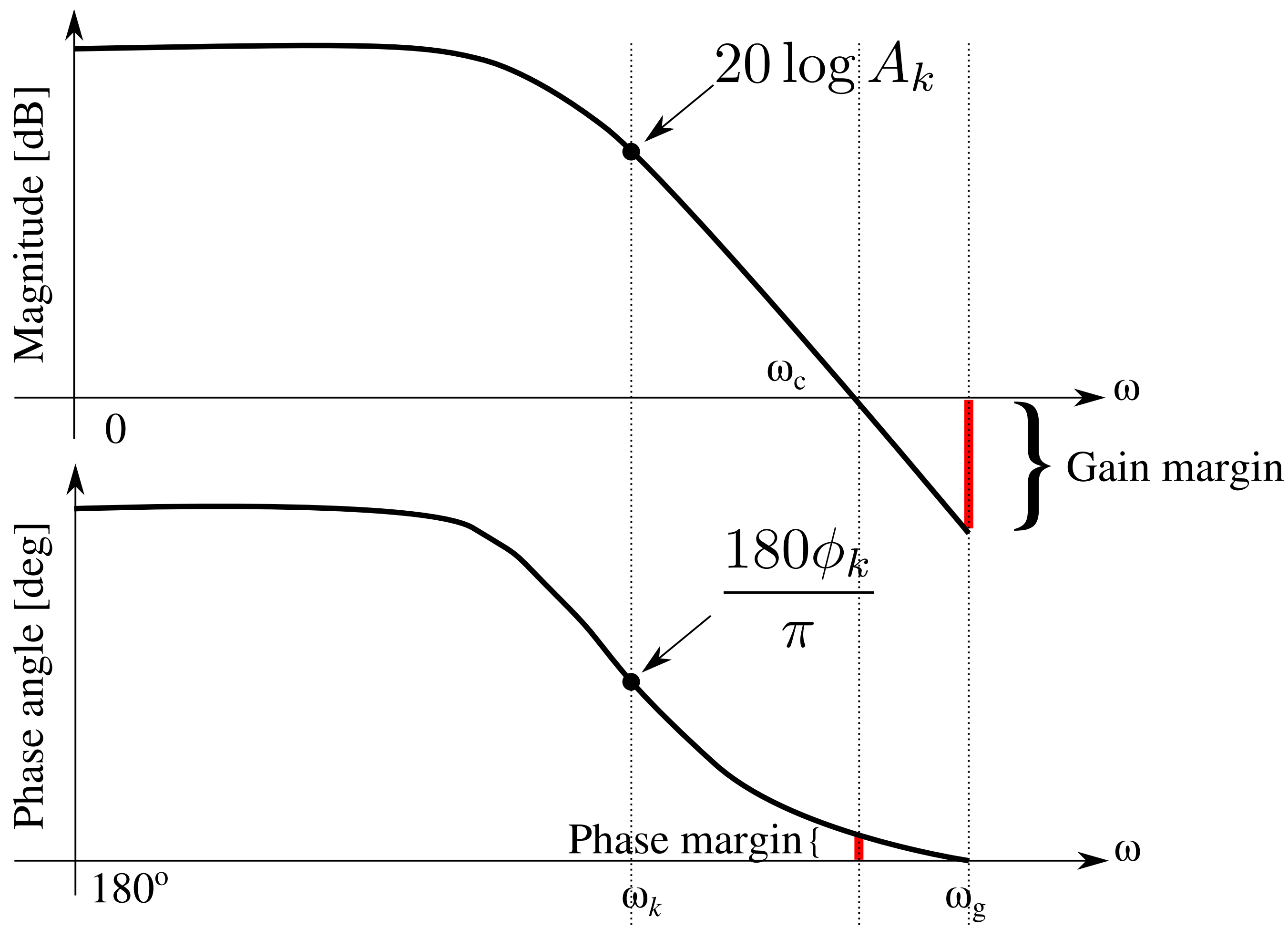
$$A_k = \left| G(j\omega_k) \right|, \quad \phi_k = \arg(G(j\omega_k)), \tag{22}$$

where $|\cdot|$ denotes the absolute value, and $\arg(\cdot)$—the argument (in radians) of the complex value $G(j\omega_k)$.

Frequency Domain Analysis of SISO Systems (continued)

- Frequency domain characteristics completely describe the behavior of a linear, time-invariant system.
- Frequency response is graphically represented in the following ways:
  - Bode plot—two separate graphs for magnitude and phase against frequency, usually on logarithmic scales;
  - Nyquist plot—a single graph depicting real vs. imaginary parts of the response covering the full frequency range;
  - Nichols plot—a single graph depicting magnitude vs. argument of the response.
- Since it is possible to assess qualitative properties of the linear system under study (e.g., relative stability margins), frequency domain analysis is essential in control design.

Frequency Domain Analysis of SISO Systems (continued): Bode Plot



Linear Systems: State Space Representation

Suppose that a system has p inputs $u_i(t)$, $i = 1, 2, \ldots, p$, and q outputs $y_k(t)$, $k = 1, 2, \ldots, q$, and there are n system states that make up a state variable vector $x = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^\mathsf{T}$. The state space expression of general dynamic systems can be written as

$$\begin{cases} \dot{x}_i = f_i(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_p), & i = 1, 2, \ldots, n, \\ y_k = g_k(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_p), & k = 1, 2, \ldots, q, \end{cases} \tag{23}$$

where $f_i(\cdot)$ and $g_k(\cdot)$ can be any nonlinear functions. For linear, time-invariant systems the state space expression is

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du, \end{cases} \tag{24}$$

where $u = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}^\mathsf{T}$ and $y = \begin{bmatrix} y_1 & y_2 & \cdots & y_q \end{bmatrix}^\mathsf{T}$ are the input and output vectors, respectively. The matrices A, B, C, and D are compatible matrices with sizes $n \times n$, $n \times p$, $q \times n$, and $q \times p$, respectively.

Definition 5. *A system* (24) *is said to be* ***asymptotically stable****, when all poles* λ_i*, i.e. roots of of* $\det(sI - A) = 0$*, where* I *is the identity matrix, have negative real parts, that is*

$$\Re(\lambda_i) < 0, \quad i = 1, 2, \ldots, n. \tag{25}$$

Definition 6. *The system* (24) *is said to be* ***completely controllable****, if it is possible to choose such input signals* $u(t)$ *that allow to move the system from state* $x(0)$ *to state* $x(t)$ *in finite time. The controllability condition is*

$$\operatorname{rank} \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} = n. \tag{26}$$

Definition 7. *The system* (24) *is said to be* ***completely observable****, if all system states are measurable from output variable values. The observability condition is*

$$\operatorname{rank} \left[\begin{bmatrix} C^T & C^T A^T & C^T \left(A^T\right)^2 & \cdots & C^T \left(A^T\right)^{n-1} \end{bmatrix}^T \right] = n. \tag{27}$$

Linear Systems: Linear Quadratic Optimal Control (LQR)

To design a state-feedback control law $u = -Kx$ we minimize the quadratic cost

$$J = \int_0^\infty \left(x^\intercal Q x + u^\intercal R u\right) \mathrm{d}t, \tag{28}$$

where Q and R are compatible design matrices, subject to $\dot{x} = Ax + Bu$ of (24). The solution is obtained by solving the Riccati equation

$$A^\intercal S + SA - SBR^{-1}B^\intercal S + Q = 0 \tag{29}$$

and taking $K = R^{-1}B^\intercal S$.

The closed-loop system becomes

$$\begin{cases} \dot{x} = (A - BK)x + Bu, \\ y = Cx + Du. \end{cases} \tag{30}$$

Questions?

Thank you for your attention!

NB! Next time we do Test #1.