ISS0031 Modeling and Identification

Exercises for Lecture 2: Formulating Linear Programming Problems

1.1: A dealer has 1500 EUR only for a purchase of rice and wheat. A bag of rice costs 150 EUR and a bag of wheat costs 120 EUR. He has a storage capacity of ten bags only and the dealer gets a profit of 11 EUR and 8 EUR per bag of rice and wheat, respectively. Formulate the problem of deciding how many bags of rice and wheat should dealer buy in order to get the maximum profit.

1.2: Mr. Bob's bakery sells bagel and muffins. To bake a dozen bagels Bob needs 5 cups of flour, 2 eggs, and one cup of sugar. To bake a dozen muffins Bob needs 4 cups of flour, 4 eggs and two cups of sugar. Bob can sell bagels in 10 EUR/dozen and muffins in 12 EUR/dozen. Bob has 50 cups of flour, 30 eggs and 20 cups of sugar. Formulate the problem of deciding how many bagels and muffins should Bob bake in order to maximize his revenue.

1.3: A company makes two types of sofas, regular and long, at two locations, one in Tallinn and one in Tartu. The plant in Tallinn has a daily operating budget of 45 000 EUR and can produce at most 300 sofas daily in any combination. It costs 150 EUR to make a regular sofa and 200 EUR to make a long sofa at the Tallinn plant. The Tartu plant has a daily operating budget of 36 000 EUR, can produce at most 250 sofas daily in any combination and makes a regular sofa for 135 EUR and a long sofa for 180 EUR. The company wants to limit production to a maximum of 250 regular sofas and 350 long sofas each day. The company makes a profit of 50 EUR on each regular sofa and 70 EUR on each long sofa. Formulate the problem of deciding how many of each type should be made at each plant in order to maximize profit.

1.4: A small company produces two types of products bacon and cheese and sells them at a profit of 4 EUR/kg and 6 EUR/kg, respectively. A student is trying to decide on lowest cost diet that provides sufficient amount of proteins and fats. He knows that bacon contains 2 units of protein/kg, 5 units of fat/kg and cheese contains 2 units of protein/kg, 3 units of fat/kg. Moreover, for the proper diet student needs to consume 9 units of protein/day and 10 units of fat/day. Formulate the problem of deciding how much student should consume of food to meet the daily norm and the cost of food was minimal.

1.5: There are m different types of food $F_1, \ldots, F_m$, that supply varying quantities of the n nutrients $N_1, \ldots, N_n$, that are essential to good health. Let b_j be the minimum daily requirement of nutrient N_j. Let c_i be the price per unit of food F_i. Let a_{ij} be the amount of nutrient N_j contained in one unit of food F_i. The problem is to supply the required nutrients at minimum cost.

1.6: There are m ports, or production plants P_i for $i = 1, \ldots, m$, that supply a certain commodity, and there are n markets M_j for $j = 1, \ldots, n$ to which this commodity must be shipped. Port P_i possesses an amount a_i of the commodity, and market M_j must receive the amount b_j of the commodity. Let c_{ij} be the cost of transporting one unit of the commodity from port P_i to market M_j. The problem is to meet the market requirements at minimum transportation cost.

1.7: There are 5 jobs that have to be given to 5 workers in such a way that each job is performed by only one worker. Since each worker can spend a certain amount of time to perform a certain task, we need to find a distribution of tasks among all workers that the total time was minimal. The table below gives the amount of time required for each worker to perform the corresponding job:

Job	Worker				
	W_1	W_2	W_3	W_4	W_5
J_1	5	4	3	6	4
J_2	3	9	8	8	4
J_3	2	1	4	5	6
J_4	3	4	2	4	3
J_5	2	6	5	3	2

1.8: A company is involved in the production of two items (I_1 and I_2). The resources need to produce I_1 and I_2 are twofold, namely machine time for automatic processing and craftsman time for hand finishing. The table below gives the number of minutes required for each item:

Item	Machine time	Craftsman time
I_1	13	20
I_2	19	29

The company has 40 hours of machine time available in the next working week but only 35 hours of craftsman time. Machine time is costed at 10 EUR per hour worked and craftsman time is costed at 2 EUR per hour worked. Both machine and craftsman idle times incur no costs. The revenue received for each item produced (all production is sold) is 20 EUR for I_1 and 30 EUR for I_2. The company has a specific contract to produce 10 items of I_1 per week for a particular customer. Formulate the problem of deciding how much to produce per week as a linear program.

1.9: A company makes two products (P_1 and P_2) using two machines (M_1 and M_2). Each unit of P_1 that is produced requires 50 minutes processing time on machine M_1 and 30 minutes processing time on machine M_2. Each unit of P_2 that is produced requires 24 minutes processing time on machine M_1 and 33 minutes processing time on machine M_2. At the start of the current week there are 30 units of P_1 and 90 units of P_2 in stock. Available processing time on machine M_1 is forecast to be 40 hours and on machine M_2 is forecast to be 35 hours. The demand for P_1 in the current week is forecast to be 75 units and for P_2 is forecast to be 95 units. Company policy is to maximize the combined sum of the units of P_1 and the units of P_2 in stock at the end of the week. Formulate the problem of deciding how much of each product to make in the current week as a linear program.

1.10: Determine two non-negative rational numbers such that their sum is maximum provided that their difference exceeds four and three times the first number plus the second should be less than or equal to 9. Formulate the problem as a linear programming problem.

Answers to problems

1. Let x_1 be rice and x_2 be wheat, then

$$z = 11x_1 + 8x_2 \to \max$$

$$\begin{aligned} 5x_1 + 4x_2 &\leq 50 \\ x_1 + x_2 &\leq 10 \\ x_1 \geq 0, x_2 &\geq 0 \end{aligned}$$

2. Let x_1 be bagels and x_2 be muffins, then

$$z = 10x_1 + 12x_2 \to \max$$

$$\begin{aligned} 5x_1 + 4x_2 &\leq 50 \\ 2x_1 + 4x_2 &\leq 30 \\ x_1 + 2x_2 &\leq 20 \\ x_1 \geq 0, x_2 &\geq 0 \end{aligned}$$

3. Let x_1 be regular sofas made in Tallinn, x_2 be long sofas made in Tallinn, x_3 be regular sofas made in Tartu, and x_4 be long sofas made in Tartu, then

$$z = 50x_1 + 70x_2 + 50x_3 + 70x_4 \to \max$$

$$\begin{aligned} 150x_1 + 200x_2 &\leq 45000 && \text{money constraint at Tallinn} \\ x_1 + x_2 &\leq 300 && \text{Tallinn sofa limit} \\ 135x_3 + 180x_4 &\leq 36000 && \text{money constraint at Tartu} \\ x_3 + x_4 &\leq 250 && \text{Tartu sofa limit} \\ x_1 + x_3 &\leq 250 && \text{regular sofa limit} \\ x_2 + x_4 &\leq 350 && \text{long sofa limit} \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 &\geq 0 \end{aligned}$$

4. Let x_1 be bacon and x_2 be cheese, then

$$z = 4x_1 + 6x_2 \to \min$$

$$\begin{aligned} 2x_1 + 2x_2 &\geq 9 \\ 5x_1 + 3x_2 &\geq 10 \\ x_1 \geq 0, x_2 &\geq 0 \end{aligned}$$

5. Let x_i be the number of units of food F_i to be purchased per day, then

$$z = c_1x_1 + c_2x_2 + \cdots + c_mx_m \to \min$$

$$\begin{aligned} a_{1j}x_1 + a_{2j}x_2 + \cdots + a_{mj}x_m &\geq b_j \\ x_i &\geq 0 \end{aligned}$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

6. Let x_{ij} be the quantity of the commodity shipped from port P_i to market M_j, then

$$z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \to \min$$

$$\begin{aligned} \sum_{j=1}^{n} x_{ij} &\leq a_i \\ \sum_{i=1}^{m} x_{ij} &\geq b_j \\ x_{ij} &\geq 0 \end{aligned}$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

7. Let x_{ij} be the ith job performed by the jth worker. Moreover, $x_{ij} = 1$ or $x_{ij} = 0$ means that whether the jth worker performs the ith job or not. Then,

$$\begin{aligned} z = 5x_{11} + 4x_{12} + 3x_{13} + 6x_{14} + 4x_{15} + \\ + 3x_{21} + 9x_{22} + 8x_{23} + 8x_{24} + 4x_{25} + \\ + 2x_{31} + x_{32} + 4x_{33} + 5x_{34} + 6x_{35} + \\ + 3x_{41} + 4x_{42} + 2x_{43} + 4x_{44} + 3x_{45} + \\ + 2x_{51} + 6x_{52} + 5x_{53} + 3x_{54} + 2x_{55} \to \min \end{aligned}$$

$$\begin{aligned} x_{11} + x_{12} + x_{13} + x_{14} + x_{15} &= 1 \\ x_{21} + x_{22} + x_{23} + x_{24} + x_{25} &= 1 \\ x_{31} + x_{32} + x_{33} + x_{34} + x_{35} &= 1 \\ x_{41} + x_{42} + x_{43} + x_{44} + x_{45} &= 1 \\ x_{51} + x_{52} + x_{53} + x_{54} + x_{55} &= 1 \\ x_{11} + x_{21} + x_{31} + x_{41} + x_{51} &= 1 \\ x_{12} + x_{22} + x_{32} + x_{42} + x_{52} &= 1 \\ x_{13} + x_{23} + x_{33} + x_{43} + x_{53} &= 1 \\ x_{14} + x_{24} + x_{34} + x_{44} + x_{54} &= 1 \\ x_{15} + x_{25} + x_{35} + x_{45} + x_{55} &= 1 \\ x_{ij} &\in \{0, 1\} \qquad \text{for } i, j = 1, \ldots, 5 \end{aligned}$$

8. Let x_1 be the number of items of I_1 and x_2 be the number of items of I_2, then

$$\begin{aligned} z = 20x_1 + 30x_2 - 10(13x_1 + 19x_2)/60 - \\ - 2(20x_1 + 29x_2)/60 = 17.1667x_1 + 25.8667x_2 \to \max \end{aligned}$$

$$\begin{aligned} 13x_1 + 19x_2 &\leq 2400 && \text{machine time} \\ 20x_1 + 29x_2 &\leq 2100 && \text{craftsman time} \\ x_1 &\geq 10 && \text{contract} \\ x_1 \geq 0, x_2 &\geq 0 \end{aligned}$$

9. Let x_1 be the number of units of P_1 produced in the current week and x_2 be the number of units of P_2 produced in the current week, then

$$z = (x_1 + 30 - 75) + (x_2 + 90 - 95) = x_1 + x_2 - 50 \rightarrow \max$$

$$\begin{aligned}
50x_1 + 24x_2 &\leq 2400 && \text{machine } A \text{ time} \\
30x_1 + 33x_2 &\leq 2100 && \text{machine } B \text{ time} \\
x_1 &\geq 75(\text{demand}) - 30(\text{initial stock}) = 45 \\
x_2 &\geq 95(\text{demand}) - 90(\text{initial stock}) = 5
\end{aligned}$$

The aim is to maximize the number of units left in stock at the end of the week.

10. Let x_1 be the first and x_2 be the second number, then

$$z = x_1 + x_2 \rightarrow \max$$

$$\begin{aligned}
x_1 - x_2 &\geq 4 \\
3x_1 + x_2 &\leq 9 \\
x_1 \geq 0, x_2 &\geq 0
\end{aligned}$$