

Algebraic Methods for Nonlinear Control Systems

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December 12, 2017

About me

Education:

- ✓ 2012 (PhD) *Polynomial methods for nonlinear control systems*, Department of Computer Control, Tallinn University of Technology
- ✓ 2008 (MSc) *Synthesis and identification of nonlinear discrete-time models for model based control*, Department of Computer Control, Tallinn University of Technology
- ✓ 2006 (BSc, cum laude) *Maximization of Profit in Case of Cobb-Douglas Production Function*, Department of Mathematics, Tallinn University

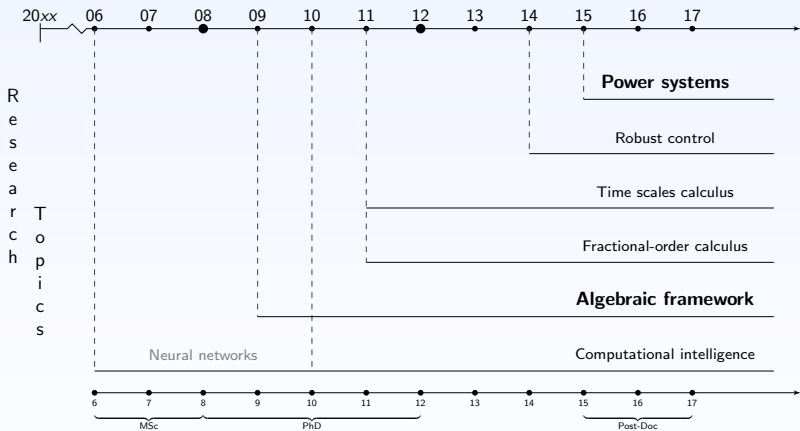
Previous positions:

- ✓ Sep. 2015 to Sep. 2017 **Postdoctoral fellow**, Faculty of Mechanical Engineering & Faculty of Electrical Engineering, Technion—Israel Institute of Technology, Israel
- ✓ Sep. 2012 to Dec. 2016 **Associate Professor**, Department of Computer Control, Tallinn University of Technology
- ✓ Aug. 2012 to Dec. 2015 **Researcher**, Institute of Cybernetics at Tallinn University of Technology
- ✓ Jan. 2010 to Aug. 2012, **Engineer**, Department of Computer Control, Tallinn University of Technology
- ✓ Nov. 2006 to Jul. 2012, **Engineer**, Institute of Cybernetics at Tallinn University of Technology

Current position:

- Jan. 2017– **Associate Professor**, Department of Computer Systems, Tallinn University of Technology

A brief history ...



Overview of the talk

- Algebraic framework: basic definitions and constructions
- Polynomial framework
- One nonlinear control problem: Realizability
- Concluding remarks

Two common theories to study nonlinear control systems

- Differential geometrical approach: appeared in the 1970s
A. Isidori, H. Nijmeijer, W. Respondek, A. van der Schaft, H. Khalil, etc.
- **Algebraic** methods of differential forms: start from the second half of 1980s
G. Conte, M. Fliess, Ü. Kotta, C. H. Moog, A. M. Perdon, etc.

Differential Algebra

Calculus and Topology:
Ordinary differentiation and
exterior derivative

Algebra:
rings, fields, etc.

Basic definitions: Calculus

Definition (Differentiability)

A real function is said to be differentiable at a point if its derivative exists at that point.

Definition (Derivative)

The derivative of a function $f(x)$ with respect to the variable x is defined as

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

Proposition

If $f(x)$ is differentiable at a point x_0 , then f is continuous at x_0 .

Example: Function $f(x) = |x|$ is continuous at 0, but not differentiable.

Analytic and meromorphic functions

Definition

Analytic function $f(x)$ is an infinitely differentiable function such that the Taylor series at any point x_0 in its domain D converges to $f(x)$ for x in a neighborhood of x_0 point-wise (and uniformly).

Examples: polynomial functions $f(x) = x^2 - 3x + 1$, exponential function $f(x) = e^x$, trigonometric functions $f_1(x) = \cos x$, $f_2(x) = \tanh(3x)$.

Definition

If I is an open subset and f is a function defined and analytic in I except for poles, then f is a meromorphic function on I .

Examples: rational functions $f(x) = \frac{x^2-1}{x^3+2x-1}$, Gamma function $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$, Riemann Zeta function $\zeta(s) = \sum_{k=1}^\infty k^{-s}$.

Analytic functions \subset Smooth functions (C^∞)

Analytic functions: more details

Definition

Let $I \subseteq \mathbb{R}$ be an open interval. A function $f : I \rightarrow \mathbb{R}$ is analytic at a point $x_0 \in I$ if it admits a Taylor series expansion in a neighborhood of x_0 . If f is analytic at every point of $I \subseteq \mathbb{R}$, we say that f is analytic in I .

Proposition

Let $I \subseteq \mathbb{R}$ be an open interval, and let $f : I \rightarrow \mathbb{R}$ be an analytic function on I , then either

- ① $f \equiv 0$ in I , or
- ② the zeros of f in I are isolated.

Non-analytic functions: Illustrative example

The function $f(x)$ defined by

$$f(x) = \begin{cases} \sin(1/x), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is not analytic because the point $x = 0$ is a point of accumulation for the zeros of f .

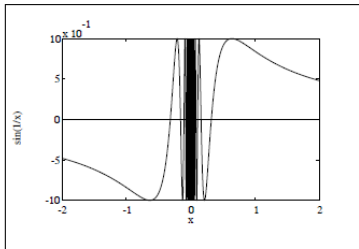


Fig. 1.2. Graph of $\sin(1/x)$

Basic algebraic structures: Ring

A **ring** is a set \mathcal{R} together with two binary operators $+$ and $*$ satisfying conditions:

- 1 **Additive associativity:** For all $a, b, c \in \mathcal{R}$, $(a + b) + c = a + (b + c)$;
- 2 **Additive commutativity:** For all $a, b \in \mathcal{R}$, $a + b = b + a$;
- 3 **Additive identity:** There exists an element $0 \in \mathcal{R}$ such that for all $a \in \mathcal{R}$, $0 + a = a + 0 = a$;
- 4 **Additive inverse:** For every $a \in \mathcal{R}$ there exists $-a \in \mathcal{R}$ such that $a + (-a) = (-a) + a = 0$;
- 5 **Left and right distributivity:** For all $a, b, c \in \mathcal{R}$, $a * (b + c) = (a * b) + (a * c)$ and $(b + c) * a = (b * a) + (c * a)$;
- 6 **Multiplicative associativity:** For all $a, b, c \in \mathcal{R}$, $(a * b) * c = a * (b * c)$ (a ring satisfying this property is sometimes explicitly termed an **associative ring**);
- 7 **Multiplicative commutativity:** For all $a, b \in \mathcal{R}$, $a * b = b * a$ (a ring satisfying this property is termed a **commutative ring**);
- 8 **Multiplicative identity:** There exists an element $1 \in \mathcal{R}$ such that for all $a \neq 0 \in \mathcal{R}$, $1 * a = a * 1 = a$ (a ring satisfying this property is termed a **unit ring**, or sometimes a **ring with identity**);
- 9 **Multiplicative inverse:** For each $a \neq 0 \in \mathcal{R}$, there exists an element $a^{-1} \in \mathcal{R}$ such that for all $a \neq 0 \in \mathcal{R}$, $a * a^{-1} = a^{-1} * a = 1$, where 1 is the identity element.

Basic algebraic structures: Summary

Prop. #	Ring	Commutative ring	Division ring / Skew field	Field
1	✓	✓	✓	✓
2	✓	✓	✓	✓
3	✓	✓	✓	✓
4	✓	✓	✓	✓
5	✓	✓	✓	✓
6	X / ✓	✓	✓	✓
7	X	✓	X	✓
8	X	✓	✓	✓
9	X	X	✓	✓

Control system

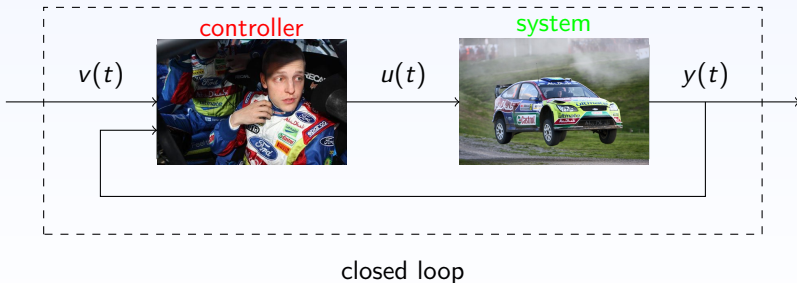
A car example

Control system

reference
desired output

input
gas pedal/wheel/etc.

output
distance/speed/etc.



Input-output and state-space forms: single-input single-output systems

Notation: the first- and second-order derivatives are $\dot{\xi} := \frac{d\xi}{dt}$, $\ddot{\xi} := \frac{d^2\xi}{dt^2}$, and $\xi^{(k)} := \frac{d^k\xi}{dt^k}$ stands to the time derivative of an arbitrary order.

Input-output equation

$$y^{(n)} = \phi\left(y, \dot{y}, \dots, y^{(n-1)}, u, \dot{u}, \dots, u^{(s)}\right).$$

State equations

$$\begin{aligned}\dot{x} &= f(x, u) \\ y &= h(x),\end{aligned}$$

$x \in \mathbb{R}^n$ is the vector of state variables,
 $u \in \mathbb{R}$ is the input signals,
 $y \in \mathbb{R}$ is the output signal,
 f and h are meromorphic functions.

Ring of analytic functions \mathcal{R}

Let \mathcal{R} denote the ring of analytic functions in a finite number of variables from the set a finite number of independent system variables from the infinite set

$$\mathcal{C}_{ss} = \{x_i, i = 1, \dots, n; u^{(k)}, k \geq 0\}$$

or

$$\mathcal{C}_{io} = \{y, y^{(1)}, \dots, y^{(n-1)}, u^{(k)}, k \geq 0\}.$$

\mathcal{C}_{ss} is associated to the state-space form

\mathcal{C}_{io} is associated to the input-output description

Differential ring

Define a time derivative operator $d/dt : \mathcal{R} \rightarrow \mathcal{R}$ as

$$\boxed{\frac{d}{dt}x = f(x, u)}, \quad \frac{d}{dt}u_j^{(k)} = u_j^{(k+1)},$$

$$\frac{d}{dt}\zeta(x, u^{(k)}) = \sum_{i=1}^n \frac{\partial \zeta}{\partial x_i} \frac{d}{dt}x_i + \sum_{k \geq 0} \frac{\partial \zeta}{\partial u^{(k)}} \frac{d}{dt}u^{(k)},$$

or as

$$\boxed{\frac{d}{dt}y^{(n-1)} = \phi(\cdot)}, \quad \frac{d}{dt}y^{(l)} = y^{(l+1)}, \text{ for } l = 0, \dots, n-2,$$

$$\frac{d}{dt}u^{(k)} = u^{(k+1)},$$

$$\frac{d}{dt}\xi(y^{(l)}, u^{(k)}) = \sum_{l=0}^{n-1} \frac{\partial \xi}{\partial y^{(l)}} \frac{d}{dt}y^{(l)} + \sum_{k \geq 0} \frac{\partial \xi}{\partial u^{(k)}} \frac{d}{dt}u^{(k)}.$$

The pair $(\mathcal{R}, d/dt)$ forms an algebraic structure known as a **differential ring**.

Differential ring: integral domain

A ring D is called an *integral domain* if it does not contain any zero divisors.

It means that if a and b are two elements of D such that $ab = 0$, then either $a = 0$ or $b = 0$ or both.

The ring \mathcal{R} of analytic functions is **integral domain**.

Differential ring: integral domain

Ring of smooth functions

Remark: C^∞ functions too form a ring, but it contains zero divisors.

Example: Consider two smooth functions defined as

$$f_1(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{if } x < 0, \\ 0, & \text{if } x \geq 0 \end{cases}$$

and

$$f_2(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ e^{-\frac{1}{x^2}}, & \text{if } x > 0, \end{cases}$$

whose product is identically zero.

Field of meromorphic functions \mathcal{K}

Construction: sketch

Construction:

- 1 Let \mathcal{S} be multiplicative subset of \mathcal{R} .
- 2 Consider the set of fractions of \mathcal{R} over \mathcal{S} , denoted as $\mathcal{K} := \mathcal{S}^{-1}\mathcal{R}$.
- 3 Elements of \mathcal{K} are meromorphic functions of the form $\beta^{-1}\alpha$, where $\alpha \in \mathcal{R}$, $\beta \in \mathcal{S}$.
- 4 Since \mathcal{R} is integral domain, \mathcal{K} forms an algebraic structure known as a **field of fractions** (quotient field).

General idea: The field of fractions \mathcal{K} of an integral domain \mathcal{R} is the smallest field containing \mathcal{R} , since it is obtained from \mathcal{R} by adding the least needed to make \mathcal{R} a field, namely the possibility of dividing by any nonzero element.

Differential field

The operator d/dt can be extended so that $d/dt : \mathcal{K} \rightarrow \mathcal{K}$.
For $b^{-1}a \in \mathcal{K}$ we define

$$\frac{d}{dt} (b^{-1}a) := (b^2)^{-1}(\dot{a}b - a\dot{b}), \quad a \in \mathcal{R}, \quad b \in \mathcal{S}.$$

The pair $(\mathcal{K}, d/dt)$ is a **differential field**.

Differential vector space \mathcal{E}

Consider next the infinite set of symbols

$$d\mathcal{C}_{ss} = \{dx_i, i = 1, \dots, n; du^{(k)}, k \geq 0\}$$

or

$$d\mathcal{C}_{io} = \{dy, dy^{(1)}, \dots, dy^{(n-1)} \mid i = 1, \dots, n; du^{(k)}, k \geq 0\}$$

and denote by \mathcal{E} the **differential vector space** spanned over the field \mathcal{K} by the elements of $d\mathcal{C}$, i.e.

$$\mathcal{E} := \text{span}_{\mathcal{K}}\{d\mathcal{C}\}.$$

Differential forms

Any element of \mathcal{E} has the form

$$\omega = \sum_{i=1}^n \alpha_i dx_i + \sum_{k \geq 0} \beta_k du^{(k)}$$

or

$$\omega = \sum_{i=1}^n \alpha_i dy^{(i)} + \sum_{k \geq 0} \beta_k du^{(k)},$$

where $\alpha_i, \beta_k \in \mathcal{K}$ and only a finite number of coefficients β_k are nonzero.

The elements of \mathcal{E} are called the differential **one-forms**.

Differential forms

Operators d and d/dt in \mathcal{E}

The differential operator $d : \mathcal{K} \rightarrow \mathcal{E}$ is defined as

$$d\zeta(x, u^{(k)}) = \sum_{i=1}^n \frac{\partial \zeta}{\partial x_i} dx_i + \sum_{k \geq 0} \frac{\partial \zeta}{\partial u^{(k)}} du^{(k)}$$

or

$$d\xi(y^{(l)}, u^{(k)}) = \sum_{l=0}^{n-1} \frac{\partial \xi}{\partial y^{(l)}} dy^{(l)} + \sum_{k \geq 0} \frac{\partial \xi}{\partial u^{(k)}} du^{(k)}.$$

For the one-form $\omega = \lambda_i d\varphi_i$, where $\lambda_i \in \mathcal{K}$ and $\varphi_i \in \mathcal{C}$, the operator $d/dt : \mathcal{E} \rightarrow \mathcal{E}$ is defined as

$$\frac{d}{dt} \left(\sum_I \lambda_I d\varphi_I \right) := \sum_I \left(\dot{\lambda}_I d\varphi_I + \lambda_I d\dot{\varphi}_I \right).$$

Remark: Operators d and d/dt commute, i.e. for $\varphi \in \mathcal{K}$

$$\frac{d}{dt}(d\varphi) = d \left(\frac{d}{dt} \varphi \right) = d\dot{\varphi}.$$

Differential forms

Example: Let $F = \sin(x_1x_2) \in \mathcal{K}$. Then differentiating F with respect to x_1 and x_2 , we get

$$dF = \cos(x_1x_2)x_2dx_1 + \cos(x_1x_2)x_1dx_2 = \cos(x_1x_2)[x_2dx_1 + x_1dx_2]$$

with $dF \in \mathcal{E}$.

Differential forms

Two-forms: exterior derivative and wedge product

Starting from the space \mathcal{E} it is possible to build up the structures used in **exterior differential calculus**. Define the set

$\wedge d\mathcal{C} = \{d\zeta \wedge d\eta \mid \zeta, \eta \in \mathcal{C}\}$, where \wedge denotes the **wedge product** with the standard properties

$$d\zeta \wedge d\eta = -d\eta \wedge d\zeta \quad \text{and} \quad d\zeta \wedge d\zeta = 0$$

for $\zeta, \eta \in \mathcal{C}$.

Introduce the space $\mathcal{E}^2 = \text{span}_{\mathcal{K}} \wedge d\mathcal{C}$ with elements being **two-forms**. The operator $d : \mathcal{E} \rightarrow \mathcal{E}^2$, called **exterior derivative** operator, is defined for $\omega = \sum_{\ell=1}^k \alpha_{\ell}(\zeta_1, \dots, \zeta_k) d\zeta_{\ell} \in \mathcal{E}$, where $\zeta_1, \dots, \zeta_k \in \mathcal{C}$, by the rule

$$d\omega := \sum_{\ell, \ell'} \frac{\partial \alpha_{\ell}}{\partial \zeta_{\ell'}} d\zeta_{\ell} \wedge d\zeta_{\ell'}.$$

Differential forms

Example: Let $\omega = dx_1 - \frac{x_1}{x_2} dx_2$, then

$$\begin{aligned} d\omega &= d\left[dx_1 - \frac{x_1}{x_2} dx_2\right] = \underbrace{d[dx_1]}_{=0} - d\left[\frac{x_1}{x_2} dx_2\right] \\ &= -\frac{\partial}{\partial x_1} \left(\frac{x_1}{x_2}\right) dx_1 \wedge dx_2 - \frac{\partial}{\partial x_2} \left(\frac{x_1}{x_2}\right) dx_2 \wedge dx_2 \\ &\qquad\qquad\qquad \underline{dx_2 \wedge dx_2 = 0} - \frac{1}{x_2} dx_1 \wedge dx_2 \end{aligned}$$

Remark: The notion of two-form can be generalized to the s -form and wedge product is defined for arbitrary s -forms.

Differential forms

Closed and exact forms

Definition

A one-form $\omega \in \mathcal{E}$ is **closed**, if $d\omega = 0$.

Definition

A one-form $\omega \in \mathcal{E}$ is **exact**, if $\omega = d\zeta$ for some $\zeta \in \mathcal{K}$.

Proposition

Any exact one-form is closed.

Differential forms

Poincaré's Lemma

Lemma (Poincaré's Lemma)

Let ω be a closed one-form in \mathcal{E} . Then there exists $\varphi \in \mathcal{K}$ such that locally $\omega = d\varphi$.

Example: Consider a closed one-form

$$\omega = \frac{x_2}{x_1^2 + x_2^2} dx_1 - \frac{x_1}{x_1^2 + x_2^2} dx_2.$$

Locally around points

- (x_1, x_2) such that $x_2 \neq 0$, we get $\omega = d[\arctan(x_1/x_2)]$;
- (x_1, x_2) such that $x_1 \neq 0$ and $x_2 = 0$, we get $\omega = d[\arctan(-x_2/x_1)]$.

However, there is no function φ such that $\omega = d\varphi$ globally.

Frobenius theorem

Definition

A subspace $\Omega \subset \mathcal{E}$ is closed or integrable, if Ω has a basis which consists only of closed forms.

Theorem

Let $\Omega = \text{span}_{\mathcal{K}}\{\omega_1, \dots, \omega_\kappa\}$. The subspace Ω is integrable if and only if

$$d\omega_i \wedge \omega_1 \wedge \dots \wedge \omega_\kappa = 0$$

for all $i = 1, \dots, \kappa$.

Frobenius theorem

Example: Consider the one-form $\omega = dx_1 + x_1 dx_2$. To verify whether ω is closed or not we need to find the exterior derivative as

$$\begin{aligned} d\omega &= d[dx_1 + x_1 dx_2] = \underbrace{d[dx_1]}_{=0} + d[x_1 dx_2] \\ &= \frac{\partial x_1}{\partial x_1} dx_1 \wedge dx_2 + \frac{\partial x_1}{\partial x_2} dx_2 \wedge dx_2 = dx_1 \wedge dx_2. \end{aligned}$$

Therefore, ω is not closed since $d\omega \neq 0$.

However, the vector space $\text{span}_{\mathcal{K}}\{\omega\}$ is integrable since

$$\begin{aligned} d\omega \wedge \omega &= dx_1 \wedge dx_2 \wedge (dx_1 + x_1 dx_2) \\ &= dx_1 \wedge dx_2 \wedge dx_1 + x_1 dx_1 \wedge dx_2 \wedge dx_2 \stackrel{dx_i \wedge dx_i = 0}{=} 0. \end{aligned}$$

Finally, if we choose the integrating factor $\alpha = 1/x_1$, then ω becomes integrable and $F = \ln|x_1| + x_2$.

Sequence \mathcal{H}_k

A sequence of subspaces

$\mathcal{H}_0 \supset \cdots \supset \mathcal{H}_{k^*} \supset \mathcal{H}_{k^*+1} = \mathcal{H}_{k^*+2} = \cdots =: \mathcal{H}_\infty$ of \mathcal{E} is defined by

$$\mathcal{H}_0 = \text{span}_{\mathcal{K}}\{dx_1, \dots, dx_n, du\},$$

$$\mathcal{H}_k = \{\omega \in \mathcal{H}_{k-1} \mid \dot{\omega} \in \mathcal{H}_{k-1}\}, \quad k \geq 1,$$

or

$$\mathcal{H}_1 = \text{span}_{\mathcal{K}}\{dy, \dots, dy^{(n-1)}, du, \dots, du^{(s)}\},$$

$$\mathcal{H}_{k+1} = \{\omega \in \mathcal{H}_k \mid \dot{\omega} \in \mathcal{H}_k\}, \quad k \geq 1.$$

Sequence \mathcal{H}_k plays an important role in the analysis of the structural properties of nonlinear systems.

Polynomial framework

Skew polynomial rings

Definition

A **skew polynomial ring** $\mathcal{A}[\partial; \alpha, \beta]$ is a noncommutative polynomial ring in ∂ with coefficients in \mathcal{A} satisfying

$$\forall a \in \mathcal{A}, \quad \partial a = \alpha(a)\partial + \beta(a).$$

Each polynomial $\pi \in \mathcal{A}[\partial; \alpha, \beta]$ can be uniquely written in the form

$$\pi = \sum_{\ell=0}^k \pi_{\ell} \partial^{k-\ell}, \quad \pi_{\ell} \in \mathcal{A}.$$

If $\pi_0 \neq 0$, then k is called the degree of π , denoted by $\deg(\pi)$.

Polynomial framework

Skew polynomial rings: special cases

Several special cases:

- **Ring of differential operators:** $\mathcal{A}[\partial; \text{id}, \frac{d}{dt}]$.
- Ring of shift operators: $\mathcal{A}[\partial; \sigma, 0]$, $\mathcal{A}[\partial; \delta, 0]$.
- Ring of difference operators: $\mathcal{A}[\partial; \tau, \tau - \text{id}]$ with $\tau a(x) = a(x + 1)$.

Definition

The skew polynomial ring, induced by $(\mathcal{K}, d/dt)$, is the ring $\mathcal{K}[\partial; \text{id}_{\mathcal{K}}, d/dt] := \mathcal{K}[\partial; d/dt]$ of polynomials with usual addition and multiplication satisfying, for any $\varsigma \in \mathcal{K} \subset \mathcal{K}[\partial; d/dt]$, the commutation rule

$$\partial \varsigma := \varsigma \partial + \dot{\varsigma}.$$

Polynomial framework

Commutation rule: examples

Example 1: Consider multiplication of two polynomials $p(\partial) = \partial^2 + 1$ and $q(\partial) = y\partial - 1$

$$\begin{aligned} p(\partial)q(\partial) &= (\partial^2 + 1)(y\partial - 1) = \partial^2 y\partial - \partial^2 + y\partial - 1 \\ &= \partial(y\partial^2 + \dot{y}\partial) - \partial^2 + y\partial - 1 \\ &= y\partial^3 + \dot{y}\partial^2 + \dot{y}\partial^2 + \ddot{y}\partial - \partial^2 + y\partial - 1 \\ &= y\partial^3 + (2\dot{y} - 1)\partial^2 + (\ddot{y} + y)\partial - 1 \end{aligned}$$

Example 2:

$$\begin{aligned} \partial \cdot (y + u + 1) &= y\partial + \dot{y} + u\partial + \dot{u}, \\ (y + u + 1) \cdot \partial &= y\partial + u\partial + \partial. \end{aligned}$$

Polynomial framework

Polynomial system description

- Recall that $\mathcal{K}[\partial; d/dt]$ is the skew polynomial ring, where ∂ is a polynomial indeterminate. Multiplication in $\mathcal{K}[\partial; d/dt]$ is defined by the commutation rule $\partial\zeta = \zeta\partial + \dot{\zeta}$, $\alpha \in \mathcal{K}$.
- Polynomial system description

$$y^{(n)} = \phi(y, \dot{y}, \dots, y^{(n-1)}, u, \dot{u}, \dots, u^{(s)})$$

$$\partial^i dy := dy^{(i)}$$

$$\partial^j du := du^{(j)}$$

$$\left[\partial^n - \sum_{i=0}^{n-1} p_i \partial^i \right] dy - \sum_{j=0}^s q_j \partial^j du = 0$$

$$p_i = \frac{\partial \phi}{\partial y^{(i)}}$$

$$q_j = \frac{\partial \phi}{\partial u^{(j)}}$$

$$p(\partial)dy + q(\partial)du = 0$$

Polynomial framework

Polynomial system description: Example

Consider the nonlinear system

$$\ddot{y} = \dot{u}y + u^2\dot{y}.$$

Define $\phi := \dot{u}y + u^2\dot{y}$ and differentiate it with respect to y, \dot{y}, u and \dot{u}

$$p_0 = \frac{\partial \phi}{\partial y} = \dot{u},$$

$$p_1 = \frac{\partial \phi}{\partial \dot{y}} = u^2,$$

$$q_0 = \frac{\partial \phi}{\partial u} = 2u\dot{y},$$

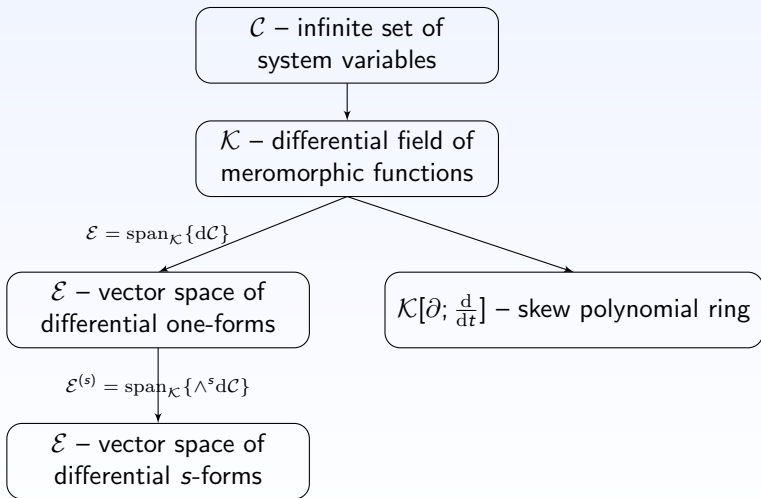
$$q_1 = \frac{\partial \phi}{\partial \dot{u}} = y.$$

Using relations $\partial^i dy := dy^{(i)}$ and $\partial^j du := du^{(j)}$, we get

$$(\partial^2 - u\partial - \dot{u})dy - (y\partial + 2u\dot{y})du = 0.$$

Algebraic and polynomial formalism: Summary

Actual picture



Problem statement

$$y^{(n)} = \phi(y, \dot{y}, \dots, y^{(n-1)}, u, \dot{u}, \dots, u^{(s)})$$

?

$$\begin{aligned}\dot{x} &= f(x, u) \\ y &= h(x)\end{aligned}$$

Goal:

Find, if possible, the state coordinates $x(t) \in \mathbb{R}^n$ such that in these coordinates the system takes the minimal state-space form.

Definition

The state-space description is said to be realization of the i/o equation if both equations have the same solution sets $\{(u(t), y(t)), t \geq 0\}$.

State of the Art

- Some of the existing results are based:
 - on the sequence of distributions of vector fields
A. J. van der Shaft, 1987
 - on the iterative Lie brackets of the vector fields
E. Delaleau and W. Respondek, 1995
 - on the sequence of the subspaces of differential one-forms
G. Conte, C. H. Moog, and A. M. Perdon, 2007
 - on polynomial framework
Ü. Kotta, M. Tönso, and J. Belikov, 2009-...
- Polynomial approach:
 - System is described by two polynomials from the skew polynomial ring
 - Solution in terms of polynomials \Rightarrow explicit formulas
 - More transparent and simple \Rightarrow easy to implement in symbolic software
 - Similar to the linear case \Rightarrow easier to understand

Realizability

Recall that the sequence of subspaces $\{\mathcal{H}_k\}_{k=1}^{\infty}$ of \mathcal{E} is defined as

$$\mathcal{H}_1 = \text{span}_{\mathcal{K}} \left\{ dy, \dots, dy^{(n-1)}, du, \dots, du^{(s)} \right\},$$
$$\mathcal{H}_{k+1} = \{ \omega \in \mathcal{H}_k \mid \dot{\omega} \in \mathcal{H}_k \}, \quad k \geq 1.$$

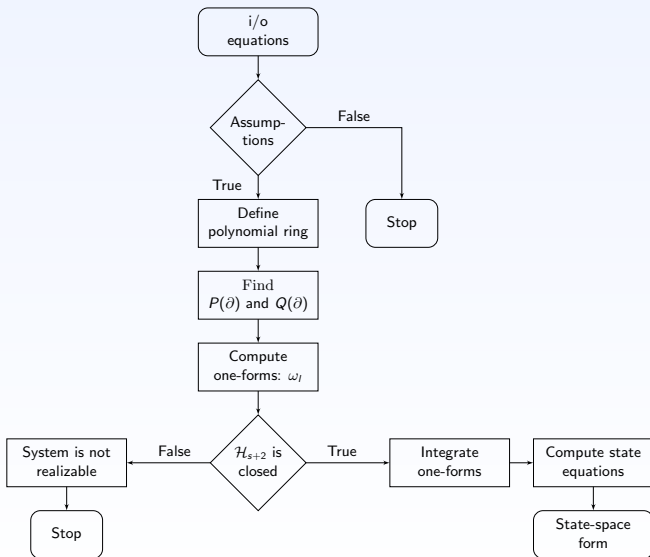
Theorem

The nonlinear i/o equation has an observable state-space realization if and only if the subspace \mathcal{H}_{s+2} is integrable.

Corollary

The state coordinates can be obtained by integrating the exact basis vectors of \mathcal{H}_{s+2} .

Realization algorithm: general idea



Computation of \mathcal{H}_{s+2} : polynomial method

Subspace \mathcal{H}_{s+2} can be calculated as

$$\mathcal{H}_{s+2} = \text{span}_{\mathcal{K}}\{\omega_1, \dots, \omega_n\},$$

where

$$\omega_l = \begin{bmatrix} p_l(\partial) & q_l(\partial) \end{bmatrix} \begin{bmatrix} dy \\ du \end{bmatrix},$$

for $l = 1, \dots, n$, and $p_l(\partial)$ and $q_l(\partial)$ can be recursively calculated from the equalities

$$\begin{aligned} p_{l-1}(\partial) &= \partial p_l(\partial) + \xi_l, & \deg \xi_l &= 0, \\ q_{l-1}(\partial) &= \partial q_l(\partial) + \gamma_l, & \deg \gamma_l &= 0 \end{aligned}$$

with the initial polynomials $p_0(\partial) := p(\partial)$ and $q_0(\partial) := q(\partial)$.

Illustrative examples

Realizable system

Consider the nonlinear i/o system

$$\ddot{y} = \dot{u}\dot{y} + uy$$

that can be described by two polynomials

$$p(\partial) = \partial^2 - \dot{u}\partial - u \quad \text{and} \quad q(\partial) = -y\partial - u.$$

Calculate two sequences of the left quotients as: $p_1(\partial) = \partial - \dot{u}$, $p_2 = 1$, and $q_1(\partial) = -\dot{y}$, $q_2 = 0$. Then, the one-forms are

$$\omega_1 = p_1(\partial)dy + q_1(\partial)du = (\partial - \dot{u})dy - \dot{y}du = d\dot{y} - \dot{u}dy - \dot{y}du,$$

$$\omega_2 = p_2(\partial)dy + q_2(\partial)du = dy,$$

and the subspace $\mathcal{H}_{s+2} = \mathcal{H}_3 = \text{span}_{\mathcal{K}}\{dy, d\dot{y} - \dot{y}du\}$ is integrable. The choice $x_1 = y$, $x_2 = e^{-u}\dot{y}$ yields the state equations

$$\dot{x}_1 = e^u x_2$$

$$\dot{x}_2 = e^{-u} u x_1$$

$$y = x_1.$$

Illustrative examples (cont.)

Non-realizable system

Consider the “ball and beam” system

$$\ddot{y} = \frac{mR^2}{J + mR^2} (y\dot{u}^2 - g \sin(u)), \quad (1)$$

where J, R, m, g are some physical parameters. The i/o equation can be described in polynomial form as

$$p(\partial) = \partial^2 - \frac{mR^2 \dot{u}^2}{J + mR^2} \quad \text{and} \quad q(\partial) = -\frac{2mR^2 y \dot{u}}{J + mR^2} \partial + \frac{gmR^2 \cos(u)}{J + mR^2}.$$

Compute the left quotients as: $p_1(\partial) = \partial, p_2(\partial) = 1$ and

$q_1(\partial) = -\frac{2mR^2}{J+mR^2} y \dot{u}, q_2(\partial) = 0$. Then, we get

$\mathcal{H}_3 = \text{span}_{\mathcal{K}}\{\omega_1, \omega_2\} = \text{span}_{\mathcal{K}}\{dy, d\dot{y} - \frac{2mR^2}{J+mR^2} y \dot{u} du\}$, which by the Frobenius theorem is not closed, since

$$d\omega_2 \wedge \omega_1 \wedge \omega_2 = \frac{2mR^2}{J + mR^2} y \dot{u} du \wedge d\dot{u} \wedge dy \wedge d\dot{y} \neq 0.$$

Therefore, the i/o equation does not admit the minimal state-space realization.

Realization: open problems

Remark:

- The realizability conditions are constructive and can be checked using \mathcal{H}_{s+2} .
- To find the state coordinates, one has to integrate the differential one-forms. The integration of (integrable in principle) differential one-forms is known to be a difficult task, in general.
- Theorem (realizability) does not define explicitly the class of i/o equations that have state-space form.

Therefore, the alternative way to tackle the realization problem is to **single out** the realizable structures for low-order i/o equations as well as to understand what can happen in case of arbitrary order, suggesting some **subclasses** of general order.

Realization: open problems

Second-order system

Consider the second-order i/o equation

$$\ddot{y} = \phi(y, \dot{y}, u, \dot{u})$$

that can be described by two polynomials

$$p(\partial) = \partial^2 - \frac{\partial\phi}{\partial\dot{y}}\partial - \frac{\partial\phi}{\partial y}$$

and

$$q(\partial) = -\frac{\partial\phi}{\partial\dot{u}}\partial - \frac{\partial\phi}{\partial u}$$

Realization: open problems

Subspace \mathcal{H}_{s+2}

Since $s = 1$, we have to check the integrability of the subspace $\mathcal{H}_3 = \text{span}_{\mathcal{K}}\{\omega_1, \omega_2\}$, where

$$\begin{aligned}\omega_1 &= p_1(\partial)dy + q_1(\partial)du = d\dot{y} - \frac{\partial\phi}{\partial\dot{u}}du, \\ \omega_2 &= p_2(\partial)dy + q_2(\partial)du = dy.\end{aligned}$$

The integrability can be checked using the Frobenius theorem, i.e. to check

$$d\omega_i \wedge \omega_1 \wedge \cdots \wedge \omega_\kappa = 0$$

for all $i = 1, \dots, \kappa$.

The first condition $d\omega_2 \wedge \omega_1 \wedge \omega_2 = 0$ is trivially satisfied.

Realization: open problems

The second condition $d\omega_1 \wedge \omega_1 \wedge \omega_2 = 0$ can be represented as

$$d \left[d\dot{y} - \frac{\partial\phi}{\partial\dot{u}} du \right] \wedge \left[d\dot{y} - \frac{\partial\phi}{\partial\dot{u}} du \right] \wedge dy = 0$$

or

$$\left[-\frac{\partial^2\phi}{\partial\dot{u}\partial\dot{y}} dy \wedge du - \frac{\partial^2\phi}{\partial\dot{u}\partial\dot{y}} d\dot{y} \wedge du - \frac{\partial^2\phi}{\partial\dot{u}\partial u} du \wedge du - \frac{\partial^2\phi}{\partial\dot{u}\partial\dot{u}} d\dot{u} \wedge du \right] \wedge \left[d\dot{y} - \frac{\partial\phi}{\partial\dot{u}} du \right] \wedge dy = 0.$$

Realization: open problems

Using the basic properties of the exterior product $d\zeta \wedge d\zeta = 0$ and $d\varepsilon \wedge d\eta = -d\eta \wedge d\varepsilon$, the above condition can be simplified as

$$-\frac{\partial^2 \phi}{\partial \dot{u} \partial \dot{u}} du \wedge d\dot{u} \wedge dy \wedge d\dot{y} = 0.$$

From the above equation, we get the partial differential equation

$$\frac{\partial^2 \phi}{\partial \dot{u} \partial \dot{u}} = 0.$$

The solutions of equation the obtained PDE define the complete subclass of the second-order i/o equations to be realizable in the state-space form. One particular solution is:

$$\phi = \phi_1(y, \dot{y}, u) + \phi_2(y, \dot{y}, u)\dot{u}.$$

Realization: open problems

Third-order system

Consider the third-order i/o equation

$$y^{(3)} = \phi(y, \dot{y}, \ddot{y}, u, \dot{u}, \ddot{u}).$$

Proceeding in the same manner as in case of the second-order i/o equation, we get the system of partial differential equations

$$\begin{cases} \phi_{\ddot{u}\ddot{u}} = 0 \\ \phi_{\ddot{u}\dot{u}} + \phi_{\ddot{u}}\phi_{\ddot{u}\ddot{y}} = 0 \\ \phi_{\dot{u}\dot{u}} - \phi_{\ddot{u}\dot{u}} + 2\phi_{\ddot{u}}\phi_{\dot{u}\ddot{y}} - \phi_{\ddot{u}}\phi_{\ddot{u}\ddot{y}} - (\phi_{\dot{u}} + \phi_{\ddot{u}}\phi_{\dot{y}})\phi_{\ddot{u}\ddot{y}} + \phi_{\ddot{u}}\phi_{\ddot{u}}\phi_{\dot{y}\ddot{y}} = 0, \end{cases}$$

where $\phi_{\alpha\beta} = \frac{\partial^2 \phi}{\partial \beta \partial \alpha}$ is used to denote the partial derivative of a function.